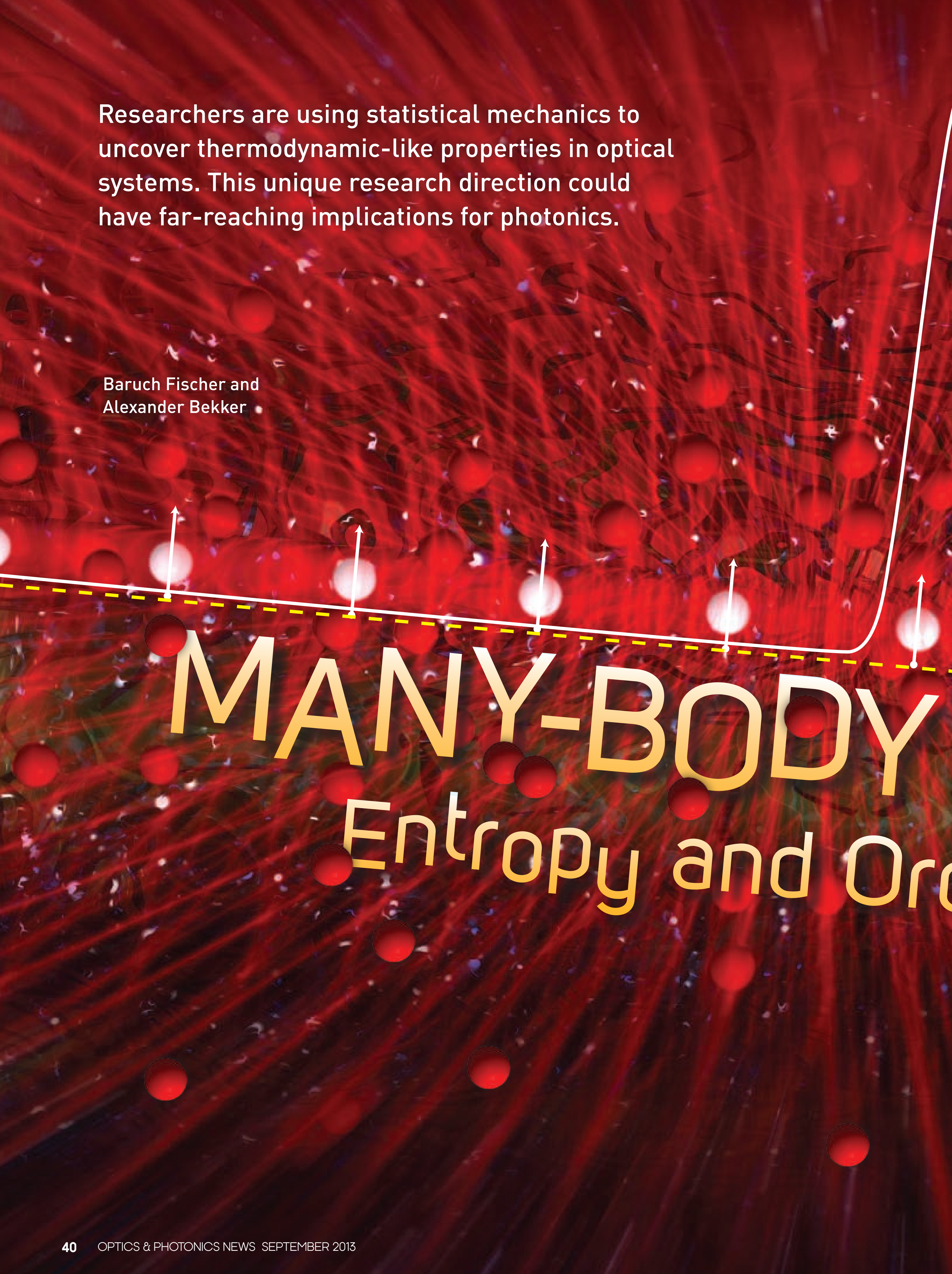


Researchers are using statistical mechanics to uncover thermodynamic-like properties in optical systems. This unique research direction could have far-reaching implications for photonics.

Baruch Fischer and
Alexander Bekker

MANY-BODY Entropy and Ord

PHOTONICS

der in a Cavity

Artist's interpretation of entropy and order in a cavity.

Illustration by Phil Saunders/spacechannel.org

In photonics, quantum mechanics gets the glory—yet its cousin statistical mechanics can also play an important role in photonic systems. Many-mode lasers are prime examples. Passive mode-locking is an exact analogue of a first-order phase transition from disordered to ordered phase and active mode-locking maps to the known spherical model, which—in higher than two dimensions—has a second-order phase transition.

Researchers first noted the similarity between second-order phase transitions and lasing shortly after the laser was invented, but they stopped short of employing many-body statistical mechanics to describe what was happening. Recently, however, there has been new interest in various many-body photonics systems, including pulsed and random lasers, as well as exciting classical and quantum light and photon condensates.

Noise is inevitable in photonic systems, and it is not just a perturbation; it is a real dimension—like temperature.

Many-body light systems

Many light systems contain many degrees of freedom. For example, lasers that generate short pulses by mode-locking can have millions of modes.

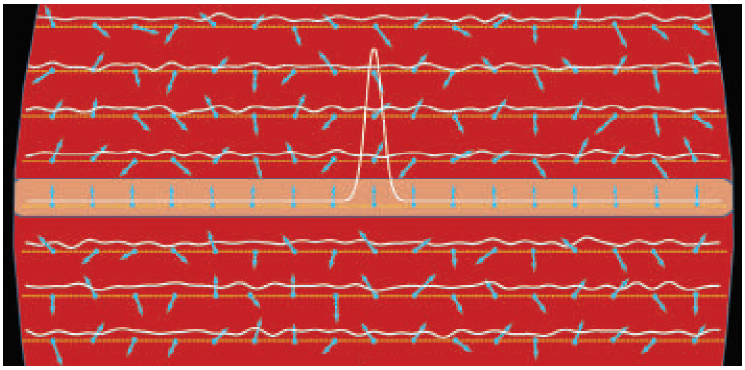
These photonic systems call for a statistical mechanical approach that was developed to treat many degrees of freedom. There is remarkable similarity, for example, between the mode-comb in the frequency domain and the canonical interacting magnetic spin systems in the spatial domain.

Mode-locked lasers include "particle" interaction akin to what exists in many-body systems, and noise takes on the role of temperature in classic systems. In addition, we talk about equilibrium rather than thermal equilibrium, since lasers are not in thermal equilibrium. With those adjustments, we see that both scenarios have a similar mathematical basis, with rigorous solutions and experimental results.

We must look at the role of entropy to understand the meaning of thermodynamic-like behavior in photonic systems. In statistical mechanics, water is not in its solid form at temperatures higher than 0 °C; here, gas or liquid phases prevail, even though the solid state has lower energy. However, a nonzero temperature elicits high-energy states that can dominate the matter state due to their very large number (entropy). It is therefore the minimum of the free energy that determines the many-body system phase. The situation is similar in passive mode-locking.

From the energetic point of view, the saturable-absorber in a cavity should drive all modes to be aligned in phase and produce short pulses right away. However, that scenario does not happen because of the small noise that evokes less favorable energies but has many-mode phasor configurations (entropy) that can dominate.

Baruch Fischer



Let There Be Ordered Light

From random mode phasors (noisy continuous wave) to ordered mode phase (pulses)

A laser cavity is full of entropy that can be viewed in the domain of modes (frequency) through the many phasor configurations (light-blue arrow arrays), or in the spatial or time domains through the many waveforms (white patterns) of light in the cavity.

In a thermodynamic model, system equilibrium is not determined by the minimum energy (H), but by the minimum *free* energy $F = H - TS$, which includes entropy (S) and temperature (T). Nonzero temperature populates high-energy states that can be numerous (entropy), and therefore this dominates the equilibrium phase.

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Noise is inevitable in photonic systems, and it is not just a perturbation; it is a real dimension—like temperature. Taking it only as a perturbation is like limiting ourselves to watching physical systems only around 0 K. The noise stems from spontaneous emission and other internal and external sources. The outcome is a competition between the energy and entropy that govern the disordered (*cw*) or ordered (pulse) mode phase separated by a first-order phase transition that is the passive mode-locking.

Statistical light-mode dynamics

Mode-locked lasers can have hundreds to millions of modes. There is clear similarity between the mode-comb in the frequency domain and the canonical interacting magnetic-spin Ising systems in the spatial domain. Instead of spins, the lasers have mode phasors, which are complex amplitudes whose directions are the phases.

The interaction between the modes due to a saturable-absorber in the cavity causes passive mode-locking (PML), and a modulation results in active mode-locking (AML). In PML, there is a long-range, four-mode interaction between all modes. In AML, it is a near-neighbor mode interaction similar to that seen in the classical Ising model for spins. In fact, AML is equivalent to the spherical model, a variant of the Ising model for magnetic spins.

The steps in the statistical light dynamic formalism in many mode lasers are:

1. The many-mode master equations are written to include a white noise term with strength T that has the role of temperature.

2. The master equations are transformed into a distribution function for the various mode configurations (using Fokker-Planck equations). We skip the detailed mathematics to give you the important bottom line: The stationary (invariant measure) mode system configurations probability function (equilibrium but not thermal equilibrium) is a Boltzmann-Gibbs-like distribution:

   $$\rho(a_0, a_{\pm 1}, \ldots) = e^{-H_I/T} / Z \,,$$

   where a_i denotes the complex mode amplitudes (phasor magnitudes and directions), T is the noise power, Z is the normalization factor, or partition function, where $Z = \Sigma\, e^{-H_I/T}$ (the summation is over all values of a_0, $a_{\pm 1}$, ...), and H_I is the Hamilonian-like energy weight responsible for the mode interaction.

   This is a central result that means that we are in the territory of statistical mechanics.

   For PML, H_I results from the saturable-absorber (with strength γ) that in the mode-space presents absorptive-based (imaginary coefficient Kerr effect) four-wave mixings between all modes: $H_I = -(\gamma/2)\, \Sigma\, a_j a^*_k a_l a^*_m$ $(j - k + l - m = 0)$.

   In AML, it is only a near neighbor interaction induced by the modulation with strength A: $H_I = -(A/2)\, \Sigma\, a_i a^*_{i\pm 1}$. We can see the similarity to magnetic spin systems, here with complex amplitudes instead of spin vectors.

3. Next comes the important and challenging step of finding Z and the free energy F in an interacting many-body system. This problem is insoluble in most statistical mechanics cases, but is exactly soluble in mode systems. PML falls in the mean field category due to the long-range interaction, and AML is mapped to the spherical model that is soluble in all dimensions.

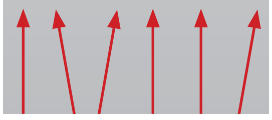
PML lasers: First-order phase transitions

In PML, the partition function Z and the free energy $F = -T \log Z$ are calculated in two ways: in the mode (frequency) domain, by computing the various mode phasors configurations and their magnitudes and directions, and in real space, by working out the various wave forms in the cavity.

Many possibilities in many-body photonics

Studies on many-mode photonic systems could lead to:

- New and **exactly soluble models** in statistical mechanics.
- Realization of **one-dimensional many-body systems** such as the laser axial mode system—a rare opportunity in physics.
- Achievement of **high-dimensional systems** such as the spherical model in any dimension (even beyond three).
- Experimental control of the **"particle" interaction** that is infrequent in statistical mechanics systems.
- Ability to **vary noise** (the photonic analog to temperature) by controlling its injection into the cavity.



For free energy, the calculation gives the following exact and very simple function:

$$F = -\frac{\gamma}{2}x^4 - T\log(P - x^2)$$

where x is the light pulse amplitude and P is the total laser power. The first term in the right hand side of F results from the energy (mode coupling) and the second from entropy. The normalized pulse amplitude expectation value $m = \langle x \rangle / P^{1/2}$ is determined by x, which gives the global minimum F. In addition, $m \leq 1$ can be considered as the system order parameter—similar to magnetization in spin systems.

We can identify in the laser two thermodynamic-like phases—one characterized by spontaneous pulses and the other by field-induced para-pulses.

For critical phenomena, we need another dimension in the phase diagram—akin to pressure in gas-liquid systems and the external magnetic field in magnetic spins. In the laser cases, it is an external pulse (comb) of strength h with a rate that exactly matches the cavity roundtrip (match the mode comb) or its multiple. It adds to the free energy F the term $-2hx$.

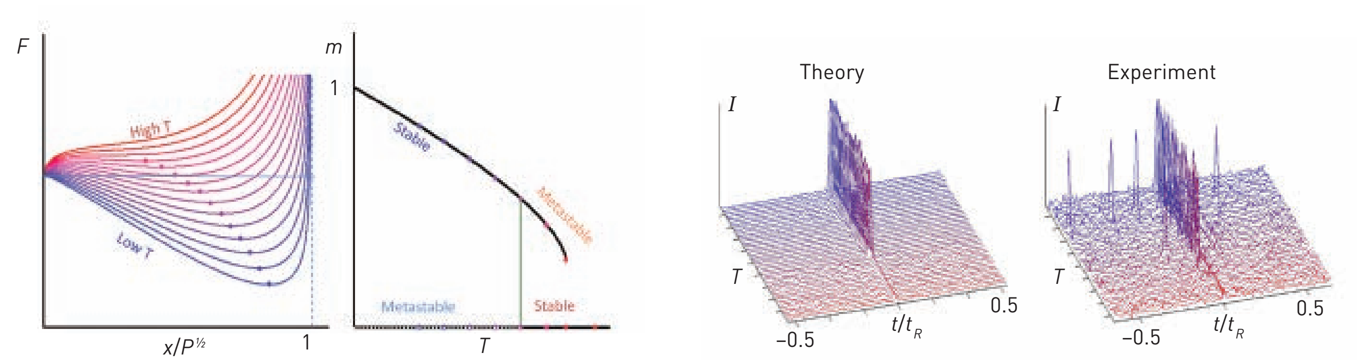
We can identify in the laser two thermodynamic-like phases—one characterized by spontaneous pulses and the other by field-induced para-pulses. They are separated by a first-order phase transition boundary that is terminated by the critical point. Such thermodynamic systems are characterized by special fingerprints called critical exponents. The experiments yield the exponents $\beta \simeq 0.5$, $\delta \simeq 3$, and $\gamma \simeq 1$, which are the mean-field values that are exact in the laser system.

AML lasers and the spherical model

In an AML system, the frequency modulation that matches the cavity resonance causes a coupling between near-neighbor modes, as in the classical Ising model. In fact, it is equivalent to the spherical model, a variant of the Ising. In the latter, the spins have values of $s_i = \pm 1$, while in the spherical model, each spin s_i can have any value, but there is a single overall constraint of $\Sigma\, s_i^2 = N$, where N is the number of spins in the system.

Similarly, in the case of the mode system, a_i are complex amplitudes (phasors) that can be of any magnitude and direction (phase), but with a constraint on the overall power of all modes: $\Sigma\, a_i a_i^* = P$. In the spherical model, only in more than two dimensions is there a second-order phase transition from disordered spin phase to long-range ordering, due to the high-dimensional connectivity. Therefore, in the usual one-dimensional AML laser, there will not be a long range phase ordering of the modes at any finite noise level.

Statistical light-mode dynamics can shed light on an inherent difference between AML and PML. The shortest pulses that can reach the regime of a few femtoseconds are obtained by passive mode locking.



PML laser with first-order phase transition

From left: Free energy (with h=0) for various noise levels; first-order phase-transition shown via the order parameter (pulse power) and light waveforms in the cavity.

PML lasers exhibit a phase transition that is absent in active ones under regular modulation, where no global mode ordering exists at any finite noise level. This falls in the category of one-dimensional many-body systems with short-range interaction.

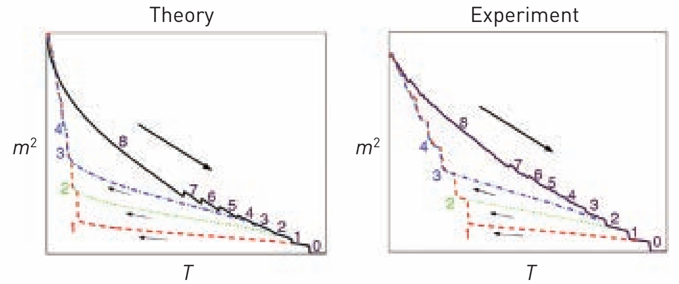
Weak noise, even spontaneous emission, can affect the phase alignment of a long but fragile mode chain (comb), preventing a global mode-phase ordering, especially for very broad frequency bandwidths. PML is different due to the saturable absorber that causes an effective long-range interaction between all modes. More complex modulations and higher dimensions can give phase transitions in AML as well—and therefore shorter pulses via condensation or the hyper-combs.

Multi-dimensional mode systems with second-order phase transitions

What about higher dimensions? A direct way to obtain high dimensional mode lattices in lasers in $\bar{k}$ space would be to use two- and three- dimensional laser cavities, which are not easily achievable with mode coupling.

We recently suggested the possibility of constructing an effective *d*-dimensional mode hyper-comb with near neighbor mode interaction from one-dimensional AML with multi-frequency modulation. Each modulation adds another dimension. The AML hyper-comb can be mapped to the spherical model in *d*-dimensions, and it can therefore serve as a rare physical realization at any dimension, even those higher than three.

One important implication is that the spherical model has in more than two dimensions, upon decreasing T/P, a second-order phase transition to a global phase-ordered mode hyper-comb. This means that hyper-combs made by AML lasers have the potential to capture very broad coherent frequency bandwidths that can generate ultimately short and robust pulses.



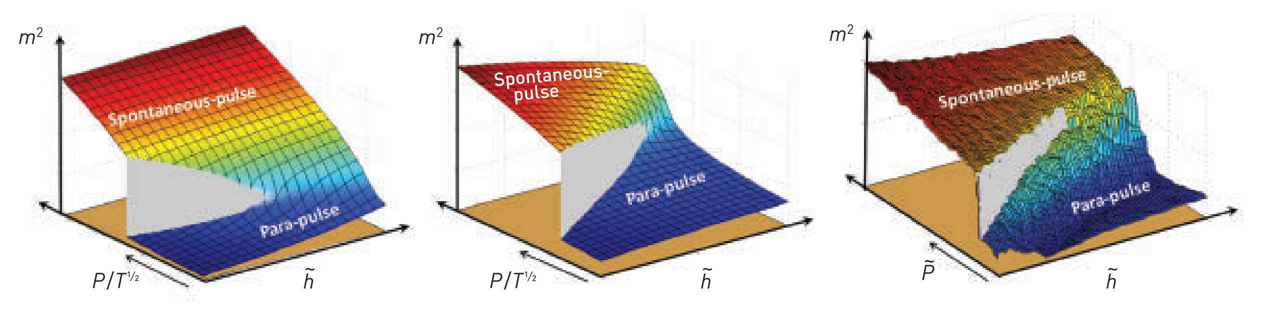
Multi-pulse generation in successive first-order phase transitions

PML in an erbium-doped fiber laser with controllable noise injection, shown by the pulse power vs. the noise power, obtained when another higher order saturation term with opposite sign is added. Phys. Rev. Lett. **93**(15), 153901 (2004).

Laser light condensation

Bose-Einstein condensation (BEC) is a special many non-interacting boson phenomenon that was observed in atomic particles at ultra-low temperatures. Researchers are increasingly turning their attention to the interesting question of whether and how a BEC can occur with non-atomic bosons, such as photons.

Here we describe two classical condensation effects in AML and cw-lasers. Both are based on weighting the modes in a noisy environment in a loss-gain scale, rather



Phase transition and critical point

Phase diagram in passive mode locking in the normalized $P/T^{1/2}$–$\tilde{h}$ plane. The first-order phase-transition line ends at a critical point. (Right) Experimental results. (Center and left) Results given by the theory; the middle one is rotated around the m^2 axis to take into account noise from the external pulse that exists in the experiment. Phys. Rev. Lett. **105**(1), 013905 (2010).

In certain conditions, AML can follow a classical route to condensation, which is seen in the lightwave pattern in the cavity in the time domain or the corresponding spatial domain.

than in the (photon) energy in BEC, and the analyses are derived directly from the master equations with noise.

Pulse condensation in AML lasers

In certain conditions, AML can follow a classical route to condensation, which is seen in the lightwave pattern in the cavity in the time domain or the corresponding spatial domain. The "particles" and "energy levels" are given by the AML eigenmodes, which, in the case of common harmonic modulation, are Hermite-Gaussian functions—where the lowest one corresponds to the optimal pulse.

We nevertheless allow for the modulation a general power-law dependence with an exponent η (formally similar to BEC in a potential trap). In most experiments, the modulation is sinusoidal and can be approximated near the lowest loss region by a quadratic dependence, $\eta = 2$ (harmonic oscillator). The modulation exponent η in AML has an important role in condensation through the density of states that they produce—just as in the case of the potential trap's exponents in BEC.

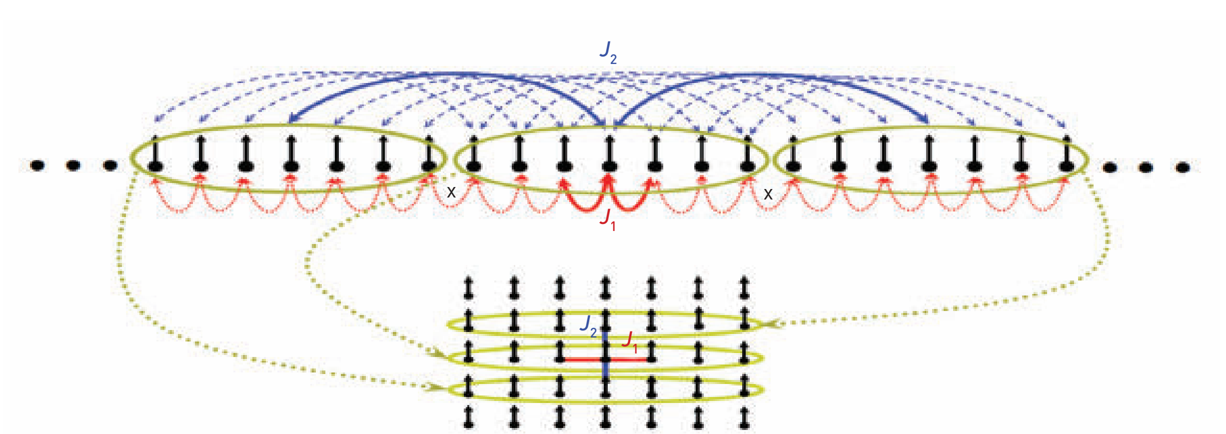
Usually the lowest loss eigenmode is the most populated state. However, noise causes a broader occupation of those states with a hierarchy that depends on their losses. Nevertheless, when the modulation exponent $\eta < 2$, the system shows a condensation route when the power increases or noise decreases—similar to the condition for BEC in a one-dimensional trap. The condensate is characterized by a sharp transition and dominance of the lowest-loss pulse eigenmode power.

CW light condensation

This condensation phenomenon is even simpler than the first. It occurs in linear mode systems such as regular cw lasers in the mode's spectral domain.

Of course, noise is present, but we also need certain conditions on the loss-gain filtering spectrum ("potential" trap). Such spectra are taken in most analytical studies to be parabolic. Here, however, we allow a general power-law dependence for the loss spectrum $\varepsilon \propto |(\omega - \omega_0)|^\eta$ near the lowest-loss mode frequency ω_0. The exponent that gives light condensation must be $\eta < 1$, compared to $\eta < 2$ in the AML case in the former section.

The light condensation behavior is similar to BEC but classical. As in BEC, there is no direct mode (particle) interaction, but rather a global constraint on the overall power ("particle" number). Nevertheless, the mode



Construction of high-dimensional mode-combs (hyper-combs)

A two-dimensional hyper-mode construction from one dimension. The active mode locking modulation has a few frequencies—including the basic one and higher orders. They induce coupling J_n between modes. The 2-D mode-lattice with nearest neighbor mode interaction is obtained by shifting mode segments, one above the other. This procedure can be repeated to higher dimensions—a rare realization of the spherical-model. Each modulation frequency adds a dimension. Opt. Express **21**(5), 6196-204 (2013).

energy levels in light condensation are measured in a loss-gain scale inherent in laser cavities, where the condensate "ground-state" is the lowest-loss mode; unlike in a BEC, that mode can be anywhere in the frequency band.

The loss scale gives a mode occupation hierarchy and power spectra that resembles Bose-Einstein distribution. Like BEC, light condensation is characterized by a negative "chemical potential" that is the gain minus the lowest loss mode value that becomes zero at condensation.

Experimental work to observe light condensation is under way.

To BEC or not to BEC: photons in optical cavities

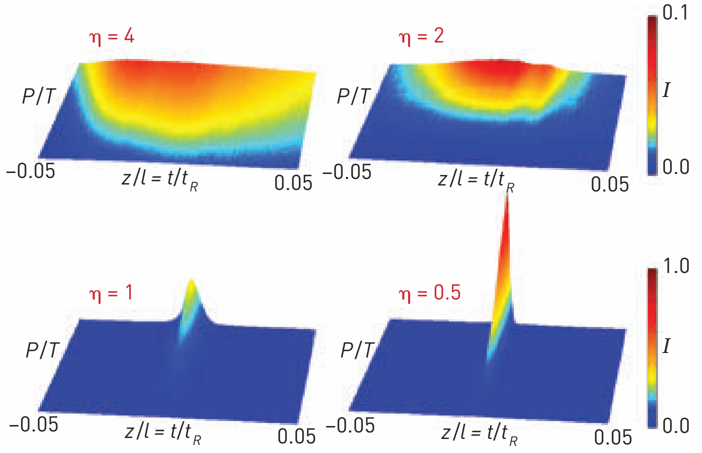
Since photons are bosons, can they exhibit real quantum-based BEC? We know that BEC necessitates the particle conservation that determines the chemical potential. Although photon gas doesn't meet that requirement and its chemical potential is zero, one can ask whether photons in high-finesse laser cavities and pumped gain media that keep the light power close to constant can show BEC. In recent work, researchers reported observing photon-BEC in a dye-filled optical microcavity at or close to room temperature. The BEC was associated with a spectrum collapse to a single frequency at the lowest transverse mode, when the power was increased beyond some critical value.

However, it's not clear whether this truly represents a thermal-quantum-based photon BEC phenomenon, since spectral collapse to a single frequency also occurs in lasing and classical light condensation. The "energy" measure of photons in multi-mode laser cavities is mostly governed by a loss-gain scale that gives the hierarchy and distribution of the modes and frequencies, rather than by photon frequency (energy) scale in thermal equilibrium.

In any case, this raises question of whether light condensation provides a new type of photon ("super-photons"?) or quantum light state.

Many directions

There's no telling where this new research area could lead—perhaps to the realization and test bed for new strictly one- or high-dimensional many-body systems and definitely to a deeper understanding of many-body photonic systems. Many-body photonics can also be approached in various ways beyond using statistical light-mode dynamics, or explored in other linear and nonlinear systems with large-scale coupled spatial modes or light channels in waveguides or free space. The opportunities are abundant. OPN



Pulse condensation in an AML laser cavity

The pulse waveform in the $z/l = t/t_R$ and $\bar{P}/T$ plane. (The variables z and l refer to cavity axis and length; t is time; and t_R is the cavity roundtrip time.) Condensation occurs for the exponents $\eta = ½, 1$. The experiment was done with an actively mode-locked erbium-doped ring fiber. Opt. Express **18**(16), 16520-5 (2010).

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ONLINE EXTRA

Visit www.osa-opn.org for videos showing pulse waveforms in PML with external injection as they vary along two paths of the phase diagram and experimental multi-pulse generation beside the corresponding order parameter in a PML laser as noise is varied.