Noise-induced oscillations in fluctuations of passively mode-locked pulses

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We study the fluctuations of pulses in mode-locked lasers using the statistical light-mode dynamics approach. The analysis is based on a decomposition of the laser waveform into three parts: solitary pulse, intracavity noise continuum, and local overlap. We discover significant features in the fluctuation dynamics, beyond those known in existing theories that disregard the continuum component of the waveform, most notably oscillations in the autocorrelation functions of the pulse power and frequency parameters, and an enhancement of the phase jitter diffusion constant. The theoretical results are corroborated by numerical simulations. © 2010 Optical Society of America

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The characterization of the statistical properties of mode-locked laser pulses, such as fluctuations or jitter in their power, frequency, timing, and phase, is important for many practical applications from communications to metrology. In a fundamental theoretical study [1], Haus and Mecozzi developed a perturbative approach to the analysis of passively mode-locked pulses in a soliton laser with strong dispersion and Kerr nonlinearity, modeled by the nonlinear Schrödinger (NLS) equation perturbed by weak saturable absorber and parabolic gain filtering terms and weak noise. In the present work we find results examining this model in view of recently developed statistical light-mode dynamics (SLD) theory that describes the many-body phenomena of the laser mode system by means of a thermodynamic analysis [2–5].

The SLD analysis has shown that the noise present in the laser cavity generates a quasi-cw disordered waveform permeating the entire cavity. The onset of mode locking is a manifestation of a thermodynamic disorder-order phase transition in the statistical steady state. Then, in a typical waveform, the laser power is divided in comparable parts between the homogeneous quasi-cw background and the narrow high-intensity pulse [6].

In the present analysis we consider the previously neglected stochastic backaction of the cw background on the mode-locked pulse. The quasi-cw waveform is a Gaussian process with a finite correlation time and thus coherently interacts with the pulse, leading to effects that are qualitatively different from those generated directly by delta-correlated driving noise, most prominently damped oscillations in the two-time correlation functions, decaying with a time scale set by the action of the saturable absorber. This feature of the pulse power evolution is reminiscent of the relaxation oscillations phenomenon observed in mode-locked lasers [7], but in our analysis we obtain that the oscillations frequency is given by the nonlinear round-trip phase accumulation of the mode-locked pulse. The nonlinear phase plays a similar role in the formation of the Kelly sidebands [8]—continuum waveforms forced by pulse perturbation whose frequency is determined by a resonance condition. Here we study the converse process, the effect of the continuum on the fluctuations in the pulse, and find analogously that it is characterized by the phase difference between the nonlinearly evolving pulse and linearly evolving continuum. Another important consequence of the pulse-cw interaction is an enhancement of the pulse power fluctuations correlation time by the ratio of the total power to the pulse power, and a corresponding enhancement of the phase diffusion coefficient by the square of this factor.

We model the laser dynamics with the NLS equation together with weak ($\mu \ll 1$) saturable absorber and spectral filtering terms, subject to the soliton condition (nonchirped pulse), and weak ($\epsilon \ll 1$) noise:

$$\dot{\psi} = (i+\mu)\left(\frac{1}{2}\psi'' + |\psi|^2\psi\right) + (g - i\phi_0)\psi + \epsilon\Gamma, \qquad (1)$$

where dot and prime symbols stand for time and space derivatives, respectively. The imaginary (dispersion and Kerr effect coefficients) in the right-hand side of Eq. (1) are normalized in the standard manner by choosing appropriate units of time, length, and power. We also follow the convention [9], of incorporating the nonlinear phase ϕ_0 directly in Eq. (1). $\epsilon\Gamma$ is a white Gaussian noise with the autocorrelation function $\langle \epsilon\Gamma_{(z_1,t_1)}\epsilon\Gamma^*_{(z_2,t_2)}\rangle = 2\epsilon^2 TL\,\delta(z_1 - z_2)\,\delta(t_1 - t_2)$, where L is the cavity length and $\epsilon^2 T$ is the total noise power injection rate.

g is the net gain coefficient; our analysis pertains only to lasers with slow saturated gain that depends on the total intracavity energy $\mathcal{P} = \int |\psi|^2 dz$. $\mathcal{P}$ is a thermodynamic quantity, and the detailed dependence of $g(\mathcal{P})$ is unimportant for the properties of the steady state. In contrast, power fluctuations in the pulse and the quasi-cw background interact with fluctuation in the gain, and these depend on the local properties of the function g near the mean value of the total power [7,10]. Here we will make the simplifying assumption that the gain is deeply saturated so that g varies very sharply in the vicinity of the mean value of $\mathcal{P}$, and the fluctuations in the total power are negligible. When the rigid power assumption is relaxed, the power fluctuations acquire an additional gain component [10].

We assume that the parameters are chosen to allow for the existence of the mode-locked pulse whose shape is approximately that of an NLS soliton ψ_p that can be described via four parameters [9]: power (energy) P, position Z_0, phase Φ, and frequency (the inverse of the group velocity) V. The steady state pulse power is P_0, and we have the relations $\phi_0 = P_0^2/8$ and $g_0 = -\mu P_0^2/8$, where $g = g_0 + \epsilon g_1$. We note that according to the chosen scaling P_0^2 gives the peak power of the pulse.

The cavity noise $\epsilon\Gamma$ generates the quasi-cw waveform $\epsilon\psi_c$ whose dynamics away from the pulse is described by the equation

$$\dot{\psi}_c = \frac{1}{2}(i + \mu)\psi_c'' + (g_0 - i\phi_0)\psi_c + \Gamma, \tag{2}$$

obtained from Eq. (1) by dropping the nonlinear terms. The decomposition of the cavity waveform into ψ_p and ψ_c is founded on the rigorous SLD analysis [4], and the two waveform parts are connected via the gain balance principle [6], which states that the steady-state pulsed laser operation is possible only if a consistent sharing of the available power with a common intracavity gain exists.

The solution of Eq. (2) is obtained by formal integration in k space [4]; the mean power carried by $\epsilon\psi_c$ is [6] $\epsilon^2 TL^2/\sqrt{2|g_0|}$ so that by the gain balance principle $P_0 = \frac{1}{2}\mathcal{P}(1 + \sqrt{1 - 8\epsilon^2 TL^2/(\mu\mathcal{P}^2)})$. Thus it becomes apparent that while both ϵ and μ are chosen small, the ratio $\epsilon/\sqrt{\mu}$ has to be of $O(1/L) \ll 1$ to sustain gain balance. This ratio also indicates the magnitude of the fluctuations in the pulse parameters relative to their steady-state values. We therefore define the following: $P = P_0 + \epsilon/\sqrt{\mu}p(t)$, $V = \epsilon/\sqrt{\mu}v(t)$, $Z_0 = \epsilon/\sqrt{\mu}z_0(t) - \int V \mathrm{d}t$, $\Phi = \epsilon/\sqrt{\mu}\phi(t) + \frac{1}{8}\int(P^2 + V^2 - P_0^2)\mathrm{d}t$.

The noise perturbation near the pulse is affected also by the nonlinear interaction with the pulse waveform, and it is not adequately described by Eq. (2). We therefore introduce the full waveform ansatz $\psi = \psi_p + \epsilon\psi_c + \epsilon\psi_1$. We emphasize that although $\epsilon\psi_c$ is small, it cannot be regarded as a perturbation, since its total energy of order $\mathcal{P}$ implies strong interactions with the pulse through the gain dynamics. On the other hand ψ_1, which is both local and small, is in effect a perturbation that can be neglected in the steady state, but is important for the pulse fluctuations.

Linearization of Eq. (1) yields for ψ_1:

$$\dot{\psi}_1 + \partial_{x_j}\psi_p \frac{\dot{x}_j}{\sqrt{\mu}} = L_1\psi_1 + L_2\psi_1^* - i\frac{1}{2}\sqrt{\mu}v\psi_p' + g_1\psi_p + f, \tag{3}$$

with implied summation over the four pulse parameters x_j. The linear operators are $L_1 = (i + \mu)(\frac{1}{2}\partial_z^2 + 2|\psi_p|^2 - P_0^2/8)$, $L_2 = (i + \mu)\psi_p^2$. The effective forcing $f = (i + \mu)(2|\psi_p|^2\psi_c + \psi_p^2\psi_c^*)$ is generated owing to nonlinearity by the overlap of ψ_p and ψ_c, and it is indeed nonnegligible only around the pulse. The gain fluctuations are obtained from the assumption of fixed total energy: $g_1\mathcal{P} = -\sqrt{\mu}P_0^2 p/4 - \int \mathrm{d}z\,\psi_p^*(\Gamma + L_1(\psi_c + \psi_1) + L_2(\psi_c^* + \psi_1^*))$.

There is a considerable freedom in the definition of ψ_1, since an $O(\epsilon)$ change in one of the pulse parameters can be absorbed into it. Here we follow a soliton perturbation theory convention, letting ψ_1 be orthogonal to the discrete right eigenspace of the linear operator acting on ψ_1 [11]. The operator defined by L_1 and L_2, which is often written as a matrix operator in the real-imaginary basis [12], breaks the full symmetry of the NLS equation and has only two zero eigenvalues, corresponding to translation (position) and phase symmetries, while its two other eigenvalues, associated with the power and frequency, are $O(\mu)$. The symmetry breaking leads to the splitting of the two two-dimensional Jordan blocks of the linear NLS operator into two pairs of nearly degenerate eigenfunctions. The linear operator acting on ψ_1 in Eq. (3) contains, in addition to L_1 and L_2, a nonlocal rank-one operator generated by the gain fluctuations term g_1. The left eigenfunctions of the full linear operator are given to $O(\mu)$ by $\chi_p = q - i\mu(1 - P_0/\mathcal{P})q$, $\chi_{z_0} = iq_z + \mu q_z$, $\chi_\phi = q - i\mu(1 - P_0/\mathcal{P})q + i2\mu(q - P_0 z q_z/2)$, $\chi_v = iq_z + \mu q_z + \mu P_0 z q/3$, where q and q_z are the right eigenfunctions of the NLS operator corresponding to phase and position, respectively [12].

The inner products with the left-hand side of Eq. (3) yield equations of motion for linear combinations of the power-phase and frequency-position coupled parameter pairs. After linear elimination we obtain

$$\dot{p} = -\mu\frac{P_0^3}{4\mathcal{P}}p - \sqrt{\mu}\int \mathrm{d}zq\,\mathrm{Re}\left[f + \frac{P_0}{\mathcal{P}}(\dot{\psi}_c - f)\right],$$

$$\dot{\phi} = \frac{2}{P_0}\sqrt{\mu}\int \mathrm{d}z\left(q - \frac{1}{2}P_0 z q_z\right)\mathrm{Im}\,f,$$

$$\dot{v} = -\mu\frac{P_0^2}{6}v - \sqrt{\mu}\int \mathrm{d}z q_z\,\mathrm{Im}\,f,$$

$$\dot{z}_0 = \frac{2}{P_0}\sqrt{\mu}\int \mathrm{d}z(zq)\mathrm{Re}\,f, \tag{4}$$

where $\Gamma + L_1\psi_c + L_2\psi_c^* = \dot{\psi}_c - f$ has been substituted in the equation for p.

As expected, the dynamics of the power and frequency has a restoring force so that they evolve to stationary processes. The statistical properties of these processes are described via the two-time correlation functions, which we calculate by substituting ψ_c into Eq. (4) and using the assumption $\mu \ll 1$ to obtain [13]

$$\langle v_{t+\tau} v_t^* \rangle = \frac{T}{P_0}\left(e^{-\mu P_0^2/6|\tau|} + \pi \int \mathrm{d}k k^2 I_k(\tau) \right),$$

$$\langle p_{t+\tau} p_t^* \rangle = \frac{2T}{P_0}\left(\frac{\mathcal{P}}{P_0} e^{-\mu P_0^3/4\mathcal{P}|\tau|} + \frac{\pi}{2} \int \mathrm{d}k I_k(\tau) \right), \quad (5)$$

with $I_k(\tau)=\mathrm{sech}^2(\pi k/2)/(k^2+1)e^{-\mu P_0^2(k^2+1)/8|\tau|}\cos(P_0^2(k^2+1)/8\tau)$.

Each of the expressions in Eq. (5) includes two terms: one decaying exponentially on a time scale equal to the strength of the restoring force in Eq. (4), and the other is a k integral term showing exponentially damped oscillation. While the first term is well known [1], being the autocorrelation of filtered white noise source, the second term results from the abiding interaction of the pulse with the cw background, which itself has a finite correlation time.

The time dependence of the oscillating terms in the p and v autocorrelation functions is shown in the left panel of Fig. 1 for $P_0=\mathcal{P}=1$ and $\mu=0.1$. The oscillations frequency is determined by the phase accumulation per round trip $P_0^2/8$, while the exponential damping with a rate given by the saturable absorber action $\mu P_0^2/8$ is augmented by k-integration induced cancellations, which amount to an additional power law damping proportional to $\tau^{-1/2}$ for large τ.

Since the position and phase correspond to exact symmetries of the system, their evolution experiences no restoring force, and their fluctuations are linear diffusion processes, driven both directly by the noise terms of Eq. (4) and by the fluctuations in frequency and power (respectively) determined from Eqs. (5). However, the direct forcing is $O(\sqrt{\mu})$ smaller than the coupling and therefore negligible. The contribution of the oscillatory part of the p and v fluctuations to the diffusion coefficients is $O(\mu)$ and therefore also negligible. As a consequence, the long-time position jitter variance is $\langle |z_0(t)|^2\rangle = 12Tt/P_0^3$, reproducing the result obtained in [1,14]. On the other hand, the phase jitter variance $\langle|\phi(t)|^2\rangle = T\mathcal{P}^2 t/P_0^3$ is larger by the factor $(\mathcal{P}/P_0)^2$ than the result in [1], which disregards the cw background. The relative enhancement of the phase fluctuations is due to the fact that the mode-locked pulse contains only a fraction of the total intracavity power, diminishing the restoring force that constrains the power fluctuations.

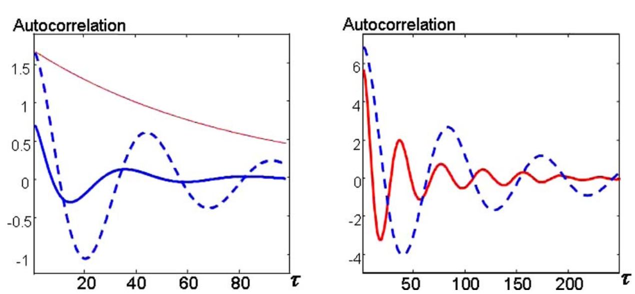

Fig. 1. (Color online) Left, theoretical analysis. Oscillatory part of power (dashed curve) and frequency (solid curve) correlations, in units of T/P_0. The thin gray (red online) curve shows the exponential damping with absorber-dependent rate. Right, numerical simulation. Power autocorrelation functions for a nonlinear coefficient of 2×10^{-3} W^{-1}, $\mu=0.1$, and steady-state pulse peak powers of 72 W (dashed, blue) and 162 W (solid red). The time scale is in round trips.

The results of our theoretical analysis are substantiated by numerical simulations of the mode-locked laser dynamics, according to Eq. (1). The noisy cw background inside the cavity was obtained from Eq. (2), and the pulse waveform was defined by subtracting it from the total waveform. The autocorrelation functions of the pulse power were calculated from the round-trip pulse power samples, and are presented for two different steady-state powers in the right panel of Fig. 1. As explained, there is a freedom in the definition of the power fluctuations, and the definition used in the theoretical analysis is different from the natural one to use in simulations [15] or experiments. Nevertheless, the qualitative properties of the fluctuations, with correlations showing weakly damped oscillations, whose frequency depends on the steady-state pulse peak power, are robust and clearly visible in the numerical results.

In conclusion, we have demonstrated that the full theoretical analysis of pulse fluctuations in mode-locked lasers must be based on the statistical steady state of the laser that includes the quasi-cw background nonperturbatively, as obtained by SLD. Two important qualitative effects are revealed by this analysis: the damped oscillatory character of the correlation functions of the power and frequency pulse parameters, and an enhancement of the phase jitter. A numerical analysis has shown that the oscillatory behavior is robust, observable for different definitions of the pulse parameters.

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