Self-starting of passive mode locking

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Abstract: It has been recently understood that mode locking of lasers has the signification of a thermodynamic phase transition in a system of many interacting light modes subject to noise. In the same framework, self starting of passive mode locking has the thermodynamic significance of a noise-activated escape process across an entropic barrier. Here we present the first dynamical study of the light mode system. While accordant with the predictions of some earlier theories, it is the first to give precise quantitative predictions for the distribution of self-start times, in closed form expressions, resolving the long standing self starting problem. Numerical simulations corroborate these results, which are also in good agreement with experiments.

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1. Introduction

Passive mode locking is a nonlinear phenomenon: It is a method where nonlinearity in the form of a saturable absorber is deliberately introduced into a laser cavity, and this nonlinearity triggers a pulsed regime in lasers. The idea is simple: an intensity fluctuation upon a uniform intensity profile will experience less loss than its surroundings, and will further grow until a pulse is formed.

However, not always does the above described process self start, and often, in order for the laser to self start, it has to operate at a sufficiently high intracavity power. Although it is plausible that a nonlinear phenomenon such as mode locking requires sufficient power, naively the equations of motion of a waveform repeatedly passing through a saturable absorber predict that pulses should always appear, only that the buildup time would scale like the inverse of the laser power, without any threshold behavior.

This puzzle has intrigued many authors in the electro-optic community, and many works were dedicated to explaining the power threshold behavior of passive mode locking and deriving a condition for self starting of pulsation. Ippen et al. [1] and later Chen et al. [2] suggested dynamic gain saturation as the source of the threshold behavior. Haus and Ippen [3] suggested reflections in the cavity to be the reason, since they introduce random dispersion to the cavity modes, and Krausz et. al. proposed [4] that a decoherence process in the cavity tends to disorder the phases of cavity modes, thus opposing mode locking. These theories have been further studied in a series of works [5, 6, 7, 8, 9], and other models have also been suggested [10, 11, 12].

Many of the above mentioned theories associate the threshold behavior with some sort of noise, randomness and decoherence. However the first *stochastic* theory of the onset of passive mode locking was presented only recently [15, 16, 17, 18, 19]. Statistical mechanics then turned

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out to be a very convenient tool for studying mode locking with noise, and the onset of pulsing in passively mode locked lasers has been described as a first order phase transition which occurs under variation of the intracavity power or of the noise power. The latter plays a role analogous to temperature in equilibrium statistical mechanics.

The resulting statistical-mechanics theory, which has been termed statistical lightmode dynamics (SLD), predicts that for high noise power or low intracavity power, cw is the stable state (in the thermodynamic sense) of a passively mode locked laser. When noise is decreased or laser power increased sufficiently, cw becomes *metastable* and the mode-locked pulsed state becomes stable, i. e., the ultimate stationary state of the laser is mode locked. However, such a metastable state can be very long lived, and the cw operation of the laser may therefore persist, even though the stable state is the mode-locked one, much as a supercooled liquid may stay unfrozen before an appropriate fluctuation drives it to the solid phase. The resulting hysteresis behavior is typical in first order phase transitions.

The self-starting problem stems therefore from the fact that the laser may remain trapped in a metastable cw state. It then needs a perturbation to drive it to the stable pulsed state, as is widely agreed. The perturbation may be provided by an external “morning wake-up kick”, but self-starting will occur only if a perturbation is provided by an internal source, i. e., by the intrinsic noise, which is also responsible for the presence of the entropic barrier which traps the laser in cw in the first place. In Fig. 1 we present a self-starting event recorded from a direct numerical simulation of the model studied in this work, see Eq. (1).

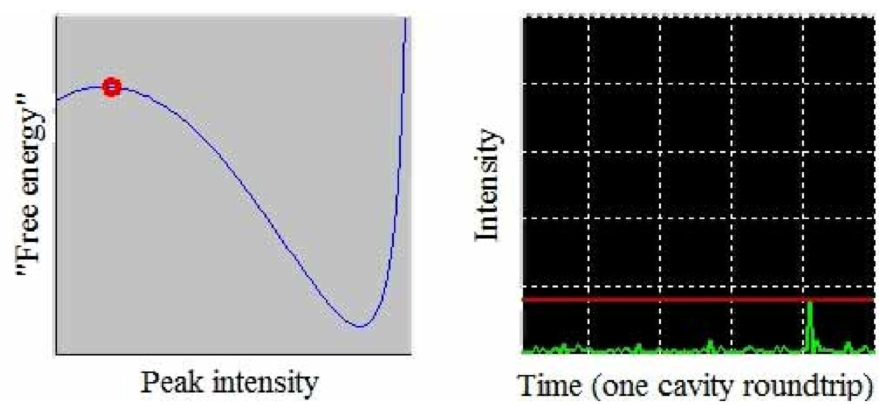

Fig. 1. Right: A recording of the time evolution of a laser waveform envelope under the action of a saturable absorber, obtained from a numerical simulation as described before. Most of the time is spent in a quiescent quasi-cw configuation of the laser, until a rare noise fluctuation quickly drives the system across the entropic barrier. Left: The horizontal position of the red dot shows the time-dependent pulse power in the simulation, and the curve shows the free energy function, which is the potential in the effective one-degree of freedom dynamics described below (`s9.avi`–1.9MB).

The self-starting process is therefore the noise-activated crossing of an entropic barrier. Since activation rates decrease exponentially with the strength of the barrier, the idea that there is a self-starting “threshold” has been put forward. In fact, when the entropic barrier is strong compared with the noise power, the self-starting process is a Poisson process, and as a consequence, the distribution of self-start times is exponential, a universal property of noise-activated escape processes [22].

In this paper we present a detailed study of the self-starting dynamics of passive mode locking, and obtain a quantitative theory of the process where a system initially in a metastable

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quasi-cw configuration reaches a steady state. We focus on the the parameter region where self starting occurs by a rare event of barrier crossing and the mean self-start time is considerably longer than the dynamical pulse buildup timescale. In the opposite case, the self-start process and rate are determined by the nonlinear dynamics of the saturable absorber and noise plays a negligible role. Since pulse buildup times are typically shorter than 1 msec, there is a broad class of passively mode locked lasers where self-starting is slowed down by a large factor because of the entropic barrier but can still be observed practically. The present theory applies to all such self starting laser systems.

The main result of this work is an explicit expression, Eq. (13), for the self-start time in a model of passive mode locking with a finite number N of modes. The number N stands for the number of active modes in the *initial* cw state, and not in the pulsed state. It is worth pointing out that the entropic barrier to mode locking exists only in such lasers, where the non mode-locked state is sometimes called "quasi-cw". Section 2 introduces this model, reviews its steady-state properties, and identifies the self-start dimensionless parameter ε, which measures the relative strength of the noise and the entropic barrier. In Sec. 3 we derive the self-start rate using the overdamped Kramers escape rate formula [21, 22], and present corroborating numerical results. The last section presents our conclusions.

2. The self-starting problem

In this work we study the onset of mode locking in a model where the spectral filtering due to the finite bandwidth of the gain is implemented by assuming that the slow electric field envelope ψ is constant over a time interval of the order of the coherence length. The physical reasoning leading to this model and its properties are discussed in Refs. [15, 20]. The model offers the simplest setting where the mode locking phase transition occurs, and as such it is natural to use it to study self-starting dynamics. Although it does not account for important dispersive terms in the dynamics, like group velocity dispersion and Kerr nonlinearity, steady state studies which include these terms [24] indicate that they have a quantitative rather than a qualitative role in the dynamics of mode locking.

The number N of constant ψ intervals is the number of active laser modes when it is operating in cw. When N is large, cavity noise creates an entropic barrier which obstructs mode locking, and may make the cw configuration stable or metastable [15]. Note that the cw operation we consider here, which is typical for multimode lasers, is characterized by the presence of many active modes with comparable amplitudes but inchoerent phases, and could be more precisely described as quasi-cw.

The master equation in our model system is

$$\dot{\psi}_m = g(t)\psi_m + \gamma_s|\psi_m|^2\psi_m + \Gamma_m(t) \ , \tag{1}$$

where ψ_m denotes the amplitude of the electric field envelope in the m-th interval, t is time, $\dot{\psi}_m$ stands for the time derivative of ψ_m, the constant γ_s is the nonlinear absorption coefficient, and g is the net gain coefficient. The cavity noise functions Γ_m are modeled by uncorrelated complex Gaussian white noise processes, with covariance

$$\begin{aligned} \langle \Gamma_m(t)\Gamma_n^*(t')\rangle &= 2W\delta_{nm}\delta(t-t') \\ \langle \Gamma_m(t)\Gamma_n(t')\rangle &= 0. \end{aligned} \tag{2}$$

We also define the mean power

$$\mathscr{P} = \frac{1}{N}\sum_{j=1}^{N}|\psi_j|^2 \ . \tag{3}$$

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Slow saturable gain is responsible for stabilizing $\mathscr{P}$ around some constant value P_0. Here we consider the simplest form of gain saturation, in which the intracavity power is kept at a strictly fixed value P_0. In this case one can obtain an explicit expression for the net gain

$$g = -\frac{1}{N\mathscr{P}}\left\{\gamma_s \sum_{j=1}^{N} |\psi_j|^4 + \mathrm{Re}\left[\sum_{j=1}^{N} \psi_j^* \Gamma_j\right]\right\}, \tag{4}$$

where the products of $\psi_j^*\Gamma_j$ are in the Stratonovich interpretation [25].

2.1. The statistical steady state

Standard methods can be used to show that the steady-state distribution ρ of the ψ_m variables under the dynamics defined by (1) with (4) is [20]

$$\rho(\psi_1,\ldots,\psi_N) \propto \delta(\mathscr{P} - P_0)\exp\left(\frac{\gamma_s}{W}\sum_j |\psi_j|^4\right). \tag{5}$$

It has been shown [20] that defining $W = NT$ and taking the limit $N \to \infty$ keeping T, γ_s, and P_0 fixed makes ρ an equilibrium distribution of a nontrivial thermodynamic system where T is the effective temperature, and the dimensionless parameter $\gamma \equiv \frac{\gamma_s P^2}{T}$ plays the role of inverse temperature. As γ increases the system exhibits a phase transition between the cw configuration and a pulsed configuration. In the cw state the intracavity power is roughly evenly divided and $|\psi_m|^2 = O(P_0)$ for all m, while in a pulsed there is a single site with power of $O(NP_0)$ and the remaining power is divided between the other sites in a statistically homogeneous manner.

It has also been shown [20] that thermodynamic quantities are exactly given by mean field theory with a pulse-power dependent free energy (Landau function) F of the form

$$F(\xi) = -\frac{1}{2}\xi^2 - \gamma \ln(\gamma - \xi), \tag{6}$$

where ξ is the pulse power divided by $\bar{p} = W/(\gamma_s P)$ (the motivation for this scaling is explained below in Sec. 2.2). In particular the steady state pulse power is equal to $\bar{p}$ times the abscissa of the global minimum of F.

The phase diagram is therefore determined by the qualitative properties of F and their dependence on γ. When $\gamma < \gamma_c = 4$, $F(\xi)$ has a single minimum at $\xi_c = 0$, corresponding to a stable cw state. When $\gamma > \gamma_c$ the minimum at zero persists, but another minimum appears at $\xi_p = \frac{1}{2}(\gamma + \sqrt{\gamma(\gamma-4)})$, corresponding to the pulsed state, along with a local maximum at $\xi_b = \frac{1}{2}(\gamma - \sqrt{\gamma(\gamma-4)})$; when $\gamma_c < \gamma < \gamma_e \approx 4.91$, $F(\xi_p) > F(\xi_c)$ and the pulsed state is metastable. The two states exchange stability at γ_e in the standard scenario of mean-field first-order phase transitions and the mode-locked state is the thermodynamically stable phase for larger values of γ. These properties of the free energy are presented graphically in Fig. (2).

It follows from these results that if a system is prepared in a pulsed state, and system parameters are changed to lower γ below γ_e, the pulsed state will persist unless a perturbation drives the system to the cw state, until the pulsed state becomes unstable when $\gamma = \gamma_c$; this hysteresis scenario is born out in experiments [26]. On the other hand, the cw state is *always* (meta)stable; it follows that the transition to the pulsed state can only occur as a crossing of the entropic barrier by a fluctuation. Self starting means that the barrier crossing is activated by an internal noise, which is also the origin of the entropic barrier. This fact is the root of the self starting problem: The lifetime of metastable states grows exponentially with the height of the barrier, and spontaneous activation will only be observed when γ is such that the strength of the barrier is not too large compared with the noise power.

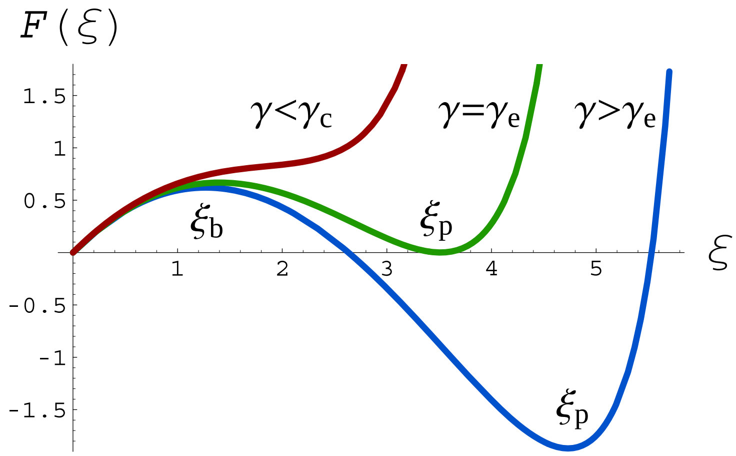

Fig. 2. The mean field free energy function (Eq. 6) and its extrema, which are also approximately the fixed points of the deterministic part of the self-start dynamics Eq. (12). The minimum at $\xi_c = 0$ corresponds to cw, the one at ξ_p corresponds to the pulsed state, and the maximum at ξ_b corresponds to the saddle of the barrier separating the cw and pulsed states.

2.2. The self-start parameter and the Arrhenius formula

The prototypical noise activated escape problem describes a particle or a system with a small number of degrees of freedom trapped in a local minimum of a confining potential and subject to thermal noise, a problem first studied quantitatively by Kramers [21]. When the energy barrier ΔE required to escape the local minimum is much larger than the temperature, escape events are very rare and the metastable lifetime t_{esc} is much longer than the typical dynamical timescale t_{dyn}. In this case, the Arrhenius formula gives the lifetime as

$$t_{esc} \sim t_{dyn} e^{\frac{\Delta E}{T}} \tag{7}$$

up to an order one factor.

The self-starting problem is of a different nature: The number N of degrees of freedom is large, while the dynamics of any particular degree of freedom is devoid of a potential barrier. The entropic barrier is exposed when one writes down the equation of motion for $p_m = |\psi_m|^2$, the power in the m-th interval, using the usual rules of stochastic calculus

$$\frac{1}{2}\dot{p}_m = \gamma_s p_m^2 (1 - \frac{p_m}{NP_0}) - \frac{W}{P_0} p_m + W - \frac{\gamma_s}{NP_0} \left(\sum_{j \neq m} p_j^2 \right) p_m + \eta_m(t) \ . \tag{8}$$

$\eta_m(t)$ is a real Gaussian white noise process derived from Γ_m, which has p_m-dependent covariance

$$\langle \eta_m(t) \eta_m(t') \rangle = W p_m (1 - \frac{p_m}{NP_0}) \delta(t - t') \ , \tag{9}$$

where now the “multiplicative noise” implied by Eq. (9) is in the Ito interpretation [25]. Note that the power of η_m vanishes at the endpoints of the allowed interval $0 \leq p_m \leq NP_0$, while the deterministic part of Eq. (8) is positive at the lower endpoint and negative at the upper endpoint; together, these two properties guarantee that p_m indeed stays in this interval.

Unlike Eq. (1), the equation for p_m contains a restoring term proportional to the intensity W of the noise. When p_m is small, the linear restoring term in Eq. (8) dominates, and p_m is trapped

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near zero, i. e. in the cw configuration. Assuming, as is argued below, that the term depending on p_j, $j \neq m$ in Eq. (8) is subdominant, escape from cw is possible only with the help of the random term η_m, derived from the cavity noise. Hence in the self-starting problem, noise is responsible both for the *existence* of the barrier, and the *activation* which allows the system to overcome the barrier. It will be shown now, however, that the barrier height depends more strongly on the noise power than the activation, so that activation is harder and cw lifetime is longer for stronger noise intensity; this state of affairs is opposite to the standard Kramers-like scenario with a given potential, where stronger noise intensity implies a shorter lifetime.

We proceed to make an estimate of the relative strength of the noise and the barrier. In order to cast this problem in a form where the usual theory of noise activated escape can be applied we rescale the variables so as to make the confining "potential" largely independent of the noise power. For this purpose let $\xi = p_m/\overline{p}$, where $\overline{p} = W/(\gamma_s P)$ is the power needed to accumulate in a single degree of freedom for the pulse buildup process to commence. This rescaling transforms Eq. (8) into

$$\xi'(\tau) = \xi^2 - \xi - \frac{\xi^3}{\gamma} + \varepsilon\left(1 - \frac{1}{NP_0^2}\left(\sum_{j\neq m} p_j^2\right)\xi\right) + \sqrt{\varepsilon\xi\left(1-\frac{\xi}{\gamma}\right)}\eta(\tau)\,, \qquad (10)$$

where $\gamma = \gamma_s P_0^2/T$ as before, $\varepsilon = \frac{\gamma}{N}$, $\tau = (W/P_0)t$ is the rescaled time variable, and the prime designates derivative with respect to τ. η is a normalized white noise, with $\langle\eta(\tau)\eta(\tau')\rangle = \delta(\tau-\tau')$.

Let us compare the deterministic drift and random diffusion terms in Eq. (10). Ignoring for a moment the terms proportional to ε, it is evident that there is an $O(1)$ negative drift for small ξ, which becomes positive for larger ξ if γ is large enough, while the activating noise is of $O(\sqrt{\varepsilon})$. It follows that the parameter ε, rather than γ, determines the character of the self starting process. If $\varepsilon = O(1)$ or larger, then the entropic barrier is too weak to inhibit the ordering interaction of the saturable absorber, and mode locking is achieved on a dynamical timescale $1/(\gamma_s P_0)$. On the other hand if $\varepsilon \ll 1$, then ξ remains trapped near zero by the drift force, and barrier crossing can occur only as a result of a rare event of a large fluctuation of η. The Arrhenius formula indicates that self-start times are then slowed down by a factor of $e^{O(\frac{1}{\varepsilon})}$. Since N is large in passively mode locked lasers, $\varepsilon = \frac{\gamma}{N}$ tends to be small even when γ is significantly larger than the threshold value needed to maintain a mode locked pulse. This observation is the basic reason that self-staring of passive mode locking difficult to achieve in many practical systems, and external perturbations are needed instead to drive these systems to mode locked operation. The case $\varepsilon \ll 1$ is the one considered in this paper. In this case the drift terms proportional to ε in Eq. (10) are small, and do not change the physical picture just described; nevertheless, because of the strong dependence of the self-start time on the drift, these small terms have an appreciable effect on the self-start process, and cannot be neglected. In the next section we carefully analyze the term proportional to $\sum_{j\neq m} p_j^2$, show that it is indeed small, and find its contribution to the self-start time.

3. The self-start time distribution

3.1. The mean field approximation

In the rest of this paper it is assumed that $\varepsilon \ll 1$. In this case self-starting occurs via noise activated barrier crossing, and the cw lifetime is exponentially larger than the other timescales in the system. Since the pulse buildup process occurs on a dynamical timescale, this fact implies that the probability of self starting occuring simultaneously in two different sites is exponentially small, and it can be safely assumed that the power at no more than a single site may

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simultaneously reach values which are much larger than the mean power P_0. Therefore the self-starting time is determined by the fastest of a system of N-independent escape processes, each described by Eq. (10).

The calculation of the mean self-start time is therefore reduced to that of a single degree of freedom ξ, governed by Eq. (10). However, this is still not a single-variable escape problem, becuase it contains the term

$$Q = \sum_{j \neq m} p_m^2 \; , \tag{11}$$

which couples the variable ξ to the $N-1$ other degrees of freedom in the system. Fortunately, in the case of interest where $\varepsilon \ll 1$, we claim that the dynamics are such that the *distribution* of Q is determined to leading order by ξ, enabling us to write a statistically equivalent equation, which involves ξ alone.

For this purpose, consider the dynamics of ψ_j with $j \neq m$. Since self-starting occurs predominantly through the growth of a single degree of freedom, it can be assumed that ψ_j remains of $O(\sqrt{P_0})$ throughout the process, and in the leading order (in ε) the nonlinear terms in its dynamics, except those involving ψ_m can be neglected. It follows that the probability distribution of ψ_j, for a *given* value of ψ_m, is gaussian. Q is therefore the sum of the fourth powers of independent centered gaussian variables, the sum of whose variances is $NP_0 - \bar{p}\xi$; from this it follows that the expectation of Q is [20] $2(\frac{NP_0 - \bar{p}\xi}{N-1})^2$. As a sum of many independent variables, the fluctuations in Q are smaller by a factor of $O(\sqrt{N})$ than its expectation, and will be neglected.

We now make the mean-field approximation by replacing Q in (10) with its expectation value conditional on ξ. The resulting equation in rescaled coordinates for the self-starting process at site m is then

$$\frac{1}{2}\xi'(\tau) = \xi^2 - \xi - \frac{\xi^3}{\gamma} + \varepsilon\left(1 - 2\left(1 - \frac{\xi}{\gamma}\right)^2 \xi\right) + \sqrt{\varepsilon\xi(1 - \frac{\xi}{\gamma})}\,\eta(\tau) \; , \tag{12}$$

3.2. The mean lifetime of cw

The self-starting problem has been reduced in Eq. (12) to a standard problem of noise-activated escape in one dimension, whose solution is well-known [25, 28]. When $\gamma > 4$, the $O(1)$ part of the drift force has three fixed points ξ_c, ξ_b and ξ_p, whose values are precisely those of the three extrema of F of Sec. 2.1. Since $\varepsilon \ll 1$, there are exact fixed points of the drift near ξ_c, ξ_b, and ξ_p, the first and last of which are (dynamically) stable, while the one near ξ_b is unstable. It follows that the drift force is derivable from a potential whose shape is well-represented by the steady-state free energy function F shown in Fig. 2, with a potential barrier near ξ_b that inhibits self-starting. The escape time may be defined as the first time the variable ξ with zero initial value reaches an arbitrary value $\bar{\xi}$ strictly between ξ_b and ξ_p. For typical escape times, which are much longer than the dynamical time scale, the distribtution of escape times is Poissonic $\Pr_{esc}(\tau) \sim e^{-\tau/\tau_{esc}}$. This fundamental property follows from the fact that the dynamics are Markovian, and from the principle of separation of scales [27]: the system reaches a quasi-steady state very fast compared to τ_{esc}, and therefore the probability that it escapes is independent of history.

The escape time τ_{esc} depends very little on the precise choice of the target pulse size $\bar{\xi}$, and is well-approximated by the mean first passage time to reach ξ. The self-start time t_{cw} is obtained from τ_{esc} by converting back to the physical variable t and dividing the result by N to take into account the fact that self start can occur via N independent escape processes. As the calculation of the mean first passage time for stochastic equations of the type of Eq. (12) is

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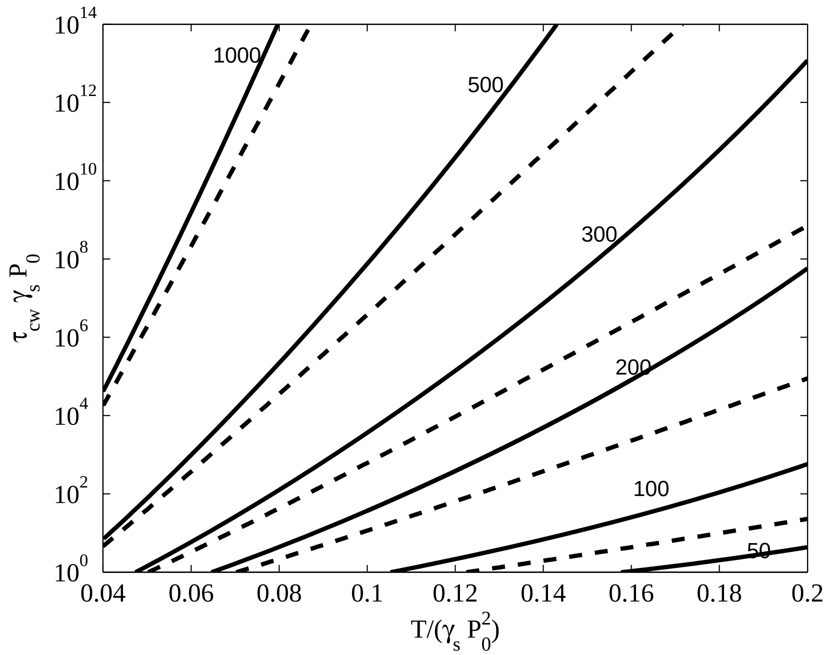

Fig. 3. A graphical comparison between the mean self-start times calculated by the asymoptotic formulas Eq. (13) (full line) and Eq. (14) (dashed line). The latter approaches the former in the lower left part of the figure. Note that rather large values of *N* are required to reach the region of validity of Eq. (14).

standard [25, 28], we merely cite the result, leaving details to the appendix

$$t_{cw} = \frac{1}{\gamma_s P_0} e^{2\xi_b - \frac{\xi_b^2}{\gamma}} \frac{\gamma - \xi_b}{\gamma^2 \xi_b} \sqrt{\frac{\pi}{2F''(\xi_b)}} \varepsilon^{5/2} e^{F(\xi_b)/\varepsilon} \,, \tag{13}$$

where $F(\xi)$ is again the free energy function. Eq. (13) is the main quantitative result of this work. It contains an exponential dependence of the self-start time on ε, which is equal to the one which can be deduced heuristically by applying the Arrhenius formula to the steady state free energy, and an algebraic prefactor, which has a $\varepsilon^{5/2}$ dependence; it is verified by numerical simulationz in Sec. 3.4.

The present analysis and its result Eq. (13) are valid when $\varepsilon \ll 1$ for any value of $\gamma > 4$; it is accompanied by a reverse process of activation from the pulsed state back to cw, which is not studied here. The steady state analysis [20] tells us that when $\gamma < \gamma_e$ the rate of activation of mode locking is slower than the rate of cw activation from the metastable mode-locked state, while the converse is true when $\gamma > \gamma_e$, in which case the process indeed describes self-starting of mode locking.

Because t_{cw} grows very fast as ε decreases, systems where self-starting may be practically observed involve ε values which are not too small; for example if $F(\xi_b)/\varepsilon$ is larger than 100, say, the probability of observing a self-starting event is less than 10^{-35} per second. Since $\varepsilon = \frac{\gamma}{N}$, and N is large, this means that γ may be assumed large in many cases. If $\gamma \gg \varepsilon^{-1}$, or

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equivalently, $\gamma \gg \sqrt{N}$ then Eq. (13) simplifies to

$$t_{cw} \sim \frac{1}{\gamma_s P_0} e^2 \sqrt{\frac{\pi}{2}} \frac{1}{\gamma} \varepsilon^{5/2} e^{\frac{1}{2\varepsilon}} \ . \tag{14}$$

Rather large values of N are needed for this asymptotic expression for the mean lifetime to become precise, as can be seen in Fig. 3, where the exact and asymptotic expressions for the mean lifetime Eqs. (13) and (14) are displayed as a function of system parameters. For reference purposes we also include a version of Eq. (14) in the original unscaled variables,

$$t_{cw} \sim \frac{1}{\gamma_s P_0} \frac{e^2 \sqrt{\pi}}{\sqrt{2}} \left(\frac{\gamma P_0^2}{T} \right)^{3/2} \frac{1}{N^{5/2}} e^{\frac{NT}{2\gamma_s P_0^2}} \tag{15}$$

3.3. Discussion and comparison with experiments

The formulation of the self-starting problem as a noise activated escape process implies that the mean self start time is an exponentially increasing function of the barrier height. Since the latter is a decreasing function of the intracavity power, a threshold-like behavior of the self-starting process has been often observed in experimental tests of self-starting with varying power.

Our theory which explains the self-starting process as an entropic barrier crossing is compatible with the idea put forward by [4, 8, 7] that the action of the saturable absorber to align the phases of the laser modes is counterbalanced by a noise-induced decoherence process, that tends to randomize them. The decoherence process and the pulse-buildup process are each characterized by an associated time scale, and self-starting occurs, according to the decoherence time theory, when the pulse-buildup time is of the order of or shorter than the decoherence time scale. The dimensionless ratio of the two time scales in our model is equal to $\gamma_s P_0^2/2W = 1/2\varepsilon$, and the condition for self starting based on the decoherence argument is therefore $\varepsilon \gtrsim 1$. As shown above, this domain is where the noise is too weak for an entropic barrier to form, and self-starting occurs on a dynamical time scale. When $\varepsilon \ll 1$ a significant entropic barrier forms, and the simple picture of decoherence versus saturable absorber-induced ordering is insufficient; rather, a full stochastic analysis is necessary leading ultimately to Eqs. (13) and (14).

The theory and results obtained here motivated a new experimental study of self-starting behavior in an additive-pulse mode locked fiber laser [23] focusing on the activation regime $\varepsilon \ll 1$, and the experimental results agree well with our theoretical predictions. Firstly, as shown in the right panel of Fig. 4, the experiments have confirmed that the distribution of the self-start times has an exponential tail, as expected in this regime. Furthermore the approximately linear relation between $\log t_{cw}$ is $1/P_0^2$ shown in the right panel of Fig. 4 agrees with our prediction in Eq. (14), which is remarkable, since the latter result has been obtained in the framework of a simplified model, and leads us to believe that this property of the self start time has a wide range of validity.

3.4. Numerical analysis

The theoretical analysis leading to the asymptotic expression for the mean lifetime Eq. (13) includes some reasoning which, while strongly supported by physical arguments, is hard to justify rigorously. We therefore performed full numerical simulations of Eq. (1) as an independent test of the theory. The numerical results, to be discussed presently below, confirm in a satisfactory manner the theoretical predictions, including the prefactor in Eq. (13).

We used the stochastic Euler method [29] and imposed the constraint (4) by normalizing $\mathscr{P}$ to P_0 in every time step. For a given value of N the problem depends on γ as a single dimensionless parameter. The initial values of the ψ_j's were chosen as independent samples a random variable with a complex Gaussian distribution and normalized to the appropriate value

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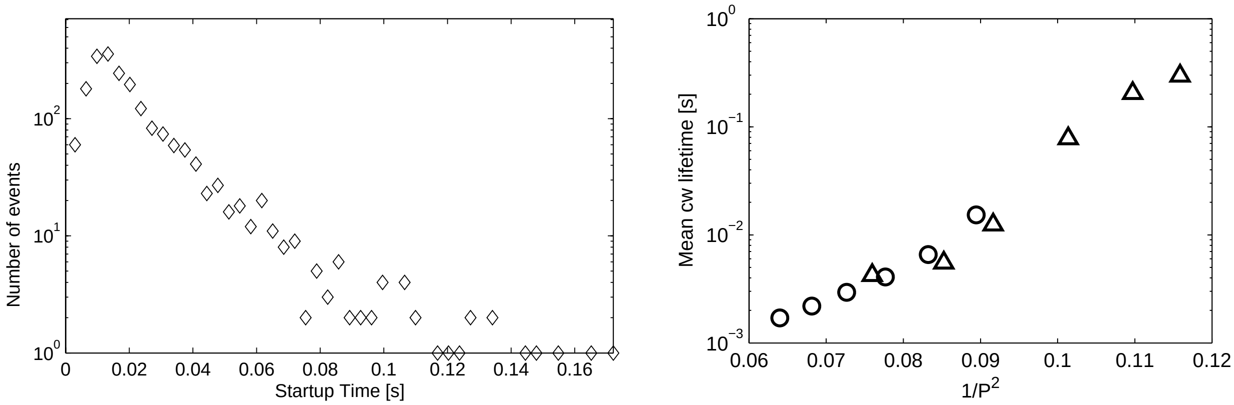

Fig. 4. Left: An experimental histogram of the measured self-start times in an additive-pulse passively mode locked fiber laser [23], shown on a logarithmic scale. An exponential tail is clearly observed, as expected for a noise-activated escape process. Right: The mean cw lifetime dependence on the intracavity power P in a set of self-starting experiments. The results are for two sets of measurements (shown in circles and triangles). The estimated mean cw lifetime is shown on logarithmic scale versus $1/P^2$. The results are compatible with the predictions following from the Arrhenius formula as discussed in the text.

of P_0. The time when the order parameter $\sum_j |\psi_j|^4$ has reached half its maximal possible value was recorded as the self-starting time.

As explained above, the theory developed in this paper has two main predictions, the first of which is that the distribution of self-start times is Poissonic beyond dynamical times, or in other words, that the probability that the system remains in the cw state decays exponentially. This prediction is confirmed by the numerical simulations, as shown in Fig. 5 where the measured probability distribution of self-start times is displayed. It is interesting to note that since this property is a consequence of the statistics of rare events, it is necessary to simulate the noise by a high-quality random number generator to obtain it correctly. In the simulations reported here we used the generator `ran2` of [30].

The second theoretical prediction concerns the mean self-start time which is given by the asymptotic approximations (13). The theoretical prediction is shown as a solid line in Fig. 6, and compared with the numerical measurements shown as red crosses. As can be seen in the figure, there is an excellent agreement between Eq. (13) and simulation, indicating that the exponential dependence on system paramaters as well as the preexponetial factor are correctly reproduced. The simulations were carried out in a parameter regime where $\gamma \sim \varepsilon^{-1}$ and Eq. (14) is not valid. Because of the $\varepsilon^{5/2}$ preexponential factor, a considerably larger number of active modes N is needed to reach the region of validity of the latter expression; on the other hand, such values of N and larger ones are quite common in actual laser systems.

4. Conclusions

The main result of this work is the theoretical demonstration that the process of self starting of passive mode locking has the dynamical significance of a noise-activated crossing of an entropic barrier, and the quantitative expression for the mean self start time. We found the dimensionless parameter that governs the process, which is equal to the the ratio of two time scales, dynamical and noise-related, in accordance with some earlier studies of self starting. Furthermore, numerical simulations corroborate the theoretical predictions with excellent accuracy, and the theory

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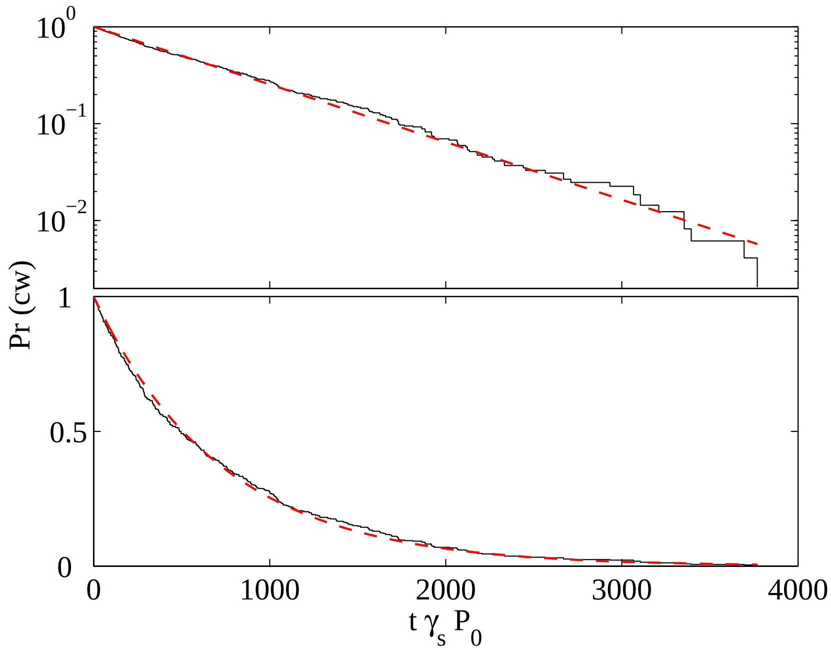

Fig. 5. A histogram of the numerically measured self-start times in logarithmic (top) and linear (bottom) scales, in a simulation performed with $N = 600$, $\gamma = 20$. The distribution is very well described by an exponential, as expected for a noise activated barrier crossing.

is also in good agreement with recent experiments in which the self-start times distribution was measured.

The self-starting parameter, which dictates the nature of the self-starting dynamics, is inversely proportional to the large number of active laser modes (in its non-pulse state). This fact is the origin of the self-starting problem, and implies that self starting is harder to achieve when the active bandwidth of the laser becomes larger.

The results presented in this paper have been obtained in the framework of a simplified model of a passively mode locked laser, where fine details of the nonlinear pulse buildup process and of the gain dyamics and its spectral filtering are not taken into account. Nevertheless, the fact that the effective activation barrier is determined by the maximal value of the static free energy divided by the self-starting parameter can serve as a good guiding principle for making an estimate of the self-starting behavior of more realistic laser systems.

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A. Appendix: Calculation of the mean self-start time

The calculation of the mean escape time for one-dimensinal Markovian activation problems such as Eq. (12) is standard [25, 28], and is outlined in this appendix for the sake of completeness. The mean first passage time $\tau_{\bar{\xi}}(\xi)$ to reach the target point $\bar{\xi}$ starting from the point ξ obeys the equation

$$\left[\xi^2-\xi-\frac{\xi^3}{\gamma}+\varepsilon\left(1-2\left(1-\frac{\xi}{\gamma}\right)^2\xi\right)\right]\frac{\partial\tau_{\bar{\xi}}(\xi)}{\partial\xi}+\varepsilon\xi\left(1-\frac{\xi}{\gamma}\right)\frac{\partial^2\tau_{\bar{\xi}}(\xi)}{\partial\xi^2}=-\frac{1}{2}\,, \qquad (16)$$

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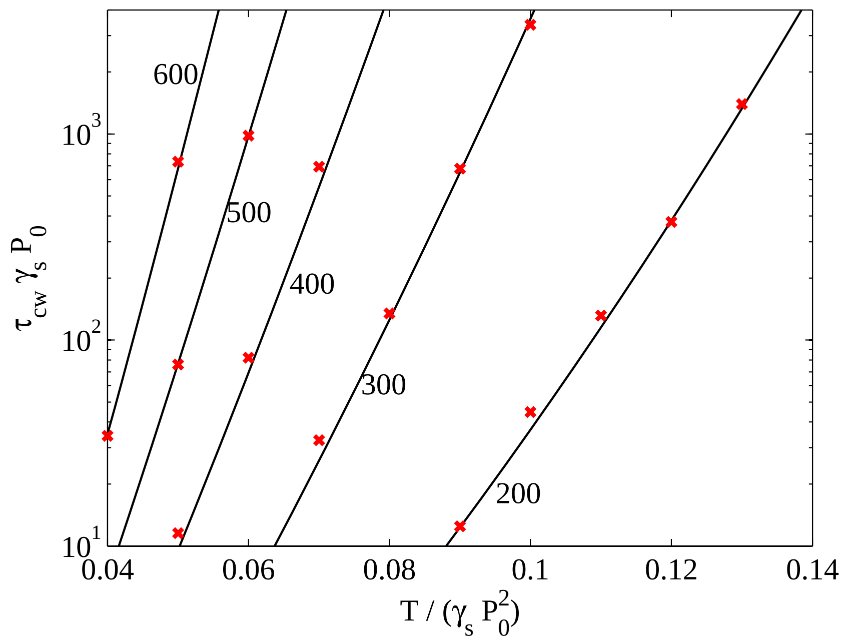

Fig. 6. A graphical comparison between the mean self-start times calculated by the asymoptotic formula Eq. (13) shown in black lines labeled by the number N of degrees of freedom, and full numerical simulation, shown with red crosses. The discrepancy between the analytically and numerically calculated lifetimes is less than 10%, and is within the statistical error.

with the boundary conditions $\tau_{\bar{\xi}}(\bar{\xi}) = 0$, and $\tau'_{\bar{\xi}}(0) = 0$ (reflecting boundary).

It is straightforward to solve this equation in quadratures for $\tau_{esc} = \tau_{\bar{\xi}}(0)$,

$$\tau_{esc} = \frac{1}{2\varepsilon}\int_0^{\bar{\xi}} d\xi \int_0^{\xi} \frac{\gamma(\gamma-\xi)}{\xi(\gamma-\xi')^2} \exp\left(\frac{1}{\varepsilon}F(\xi) - F(\xi') + 2(\xi-\xi') - \frac{\xi^2-(\xi')^2}{\gamma}\right) d\xi' \,, \quad (17)$$

where the function F is the steady state free energy defined in Sec. 2.1.

In the case of interest, $\varepsilon \ll 1$, the main contribution to the integral arises from the vicinity of the maximum of $F(\xi) - F(\xi')$ in the region $0 \le \xi' \le \xi \le \bar{\xi}$, which is reached when $\xi = \xi_b$, the abcissa of the potential barrier of F, and $\xi' = 0$. The leading term in the asymptotic expansion of τ_{esc} in powers of ε is obtained by approximating the integrand as a function quadratic in ξ and linear in ξ' near the maximum (Laplace's method), giving

$$\tau_{esc} = e^{2\xi_b - \frac{\xi_b^2}{\gamma}} \frac{\gamma-\xi_b}{\gamma\xi_b} \sqrt{\frac{\pi}{2F''(\xi_b)}}\, \varepsilon^{1/2} e^{\frac{1}{\varepsilon}F(\xi)} (1+o(1)) \,. \quad (18)$$

Eq. (13) is obtained from this result upon scaling back to the physical time variable t and division by N to take into account the N possible self-starting paths.

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