Non-Gibbsian Stochastic Light-Mode Dynamics of Passive Mode Locking

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We study a stochastic light-mode system with non-Gibbsian steady state statistics, unravelling global nonequilibrium phase transition properties. It relates to the onset of passive mode-locking in the general case of lasers with arbitrary dispersion and Kerr nonlinearity that includes the nonsolitonic regime. The solution is facilitated by a special stationarity criterion imposed by the system gain balance. We show that the mode-locking phase transition is generic, and give exact expressions for the pulse power and its stability map. We find that at the boundary of the mode-locking stability the pulse power is *exactly* one half of the total intracavity power, and that the parameter region for the most resistant pulses against noise destabilization is not at the soliton condition.

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The physics of nonequilibrium systems is an intensive research topic that touches many different areas with various meanings and examples [1]. In this Letter we analyze and solve a non-Gibbsian nonequilibrium light-mode system, realized in passive mode-locked lasers [2]. It is a special many-body system that for certain parameters was shown to be governed by equilibrium statistical mechanics, resulting in very rich thermodynamiclike behavior [3–5]. For the broader case, however, the steady state statistics can be non-Gibbsian, and therefore cannot be directly analyzed by standard equilibrium statistical mechanics methods. Here we solve the general nonequilibrium light-mode system by applying a special statistical stationarity criterion about common sharing of energy resources (gain) between the noisy and ordered waveforms in the system. The statistical steady state of far from equilibrium systems is characterized by steady probability currents and entropy production, and there is no widely applicable analysis method. Therefore the new method presented here for analyzing a class of nonequilibrium systems can be of importance. Moreover, the results provide a global basis for laser mode locking as a nonequilibrium phase transition and thus are significant to the fields of ultrashort optics and laser physics.

Passively mode-locked laser systems are centrally located in the research and technology of ultrashort optical pulses. They are the enabling means for providing light pulses that nowadays reach the few femtosecond regime, close to the limit of one lightwave cycle [6]. Transition to pulse mode-locked operation is abrupt and requires a threshold power, below which the laser operates as continuous wave (cw). This cw to pulse transition has been the subject of many studies [7–9], but only recently has it been satisfactorily resolved. The new theory is based on the observation that a multimode laser is a many-body dynamical system of interacting light modes, subject to external noise. The study of multimode lasers in this point of view has led to the development of the statistical light-mode dynamics (SLD) [3,10], in which the physics of the laser as a many-body system acquires a *thermodynamic* meaning. The SLD systems were shown to be solvable using an exact mean-fieldlike theory. The onset of mode locking is then naturally interpreted as a first order phase transition, and all the known phenomenology of passive mode locking is recovered.

The aforementioned SLD systems share a common simplifying property, known in stochastic analysis as the potential condition (detailed balance), in the steady state [11]. Then the Fokker-Planck equations are straightforwardly integrated to yield a Gibbs measure. In the context of passive mode locking, the potential condition is not met (there is a nonzero probability current) when the dispersive terms in the master equation, resulting from the chromatic dispersion and the Kerr nonlinearity, are included and they do not obey the soliton condition [3]. This condition is a constraint which relates the dispersive coefficients with the gain terms of the equation in a manner specified below. However, in applications of ultafast optics the dispersive effects are usually strong [2], thus have to be included [2,12,13], and in many cases it is not possible or convenient to achieve the soliton condition.

In this Letter we address the SLD of passive mode locking with unrestricted Kerr nonlinearity and chromatic dispersion. As said above, in the nonsolitonic case the steady state measure is not known and equilibrium methods are no longer applicable. Nevertheless, we have found a way to circumvent the problem by exploring the statistics of the order parameter directly. It was facilitated by developing a stationarity criterion to determine the pulse power through *gain balance*. Assuming that the steady state waveform decomposes into pulse and noise continuum parts, we show that the fact that both parts experience the same gain fully determines the mode-locking threshold and the pulse power. A practical outcome of our theory is a global closed form expression for the stability threshold of passive mode locking. Intriguingly, we find that the optimal parameter regime for mode locking is *not* solitonic. Another interesting result is that *the pulse power at the* *threshold is always exactly one half of the total intracavity power*, regardless of the values of other system parameters.

We model the temporal evolution of the laser waveform by the Haus master equation [2,14]. When dispersive effects are taken into account, the master equation is a complex Ginzburg-Landau equation for ψ, the slowly varying amplitude of the electric field,

$$\frac{\partial \psi}{\partial t} = A\frac{\partial^2 \psi}{\partial z^2} + B|\psi|^2\psi + g\psi + \Gamma(z, t), \qquad (1)$$

in a cavity of length L. The real and imaginary parts of the complex coefficient $A = \gamma_g + i\gamma_d$ model the real and imaginary parts of the gain parabolic spectral filtering and the chromatic dispersion, respectively, while $B = \gamma_s + i\gamma_k$ accounts, in its real and imaginary parts, for the fast saturable absorber and the Kerr nonlinearity, respectively. We also define and later use the dimensionless parameters $\mu = \gamma_k/\gamma_s$ and $\rho = \gamma_d/\gamma_g$. g is the overall net gain in the cavity, assumed to be slow relative to the time scale of variations of ψ. The slow saturable gain is modeled by letting g be a function of the total intracavity power $\frac{1}{L}\int|\psi|^2dz$; it acts as a restoring force that stabilizes the intracavity power around some value P determined by the amplifier. It follows from the analysis below that once the value of P is given, the precise functional form of g is unimportant [3,10]. The noise in the cavity, Γ, is white Gaussian with autocorrelation $\langle\Gamma^*(z', t')\Gamma(z, t)\rangle = 2LT\delta(z - z')\delta(t' - t)$, where T is the noise power injection rate.

When A/B is real (i.e., $\mu = \rho$) the master equation is said to satisfy the *soliton condition*, under which its noiseless version has solitonlike pulse solutions. Then [3] the master equation satisfies the potential condition [11], implying that the steady state waveform probability distribution is Gibbs-like. Analysis in this case can therefore be done by the powerful tools of equilibrium statistical mechanics, and the steady state mode-locking properties are obtained by calculating the free energy.

In the general case where the soliton condition does not hold, the master equation does not obey the potential condition, and the steady state distribution is unknown. However, the steady state can be studied directly from the master equation by the following mean-field-like argument: in the thermodynamic limit, waveform configurations which have a significant probability in the steady state distribution are either a combination of a narrow pulse (ψ_s) which carries a macroscopic fraction of the intracavity power with a noisy cw background (ψ_n), or pure cw configurations. This hypothesis was rigorously established in the soliton case [10]. It is also supported by numerical analysis as we show below.

Since the pulse peak power is much larger than the mean power P, the decomposition scheme implies that the dynamics of the pulse is governed mainly by the deterministic terms in Eq. (1) [14]. The continuum background has a much weaker amplitude, therefore the nonlinear term in Eq. (1) does not contribute significantly to its dynamics. Furthermore, the pulse occupies only a very small fraction of the cavity length, and therefore the contribution of the pulse waveform fluctuations to the entropy is negligible in the thermodynamic limit. The ψ fluctuations are therefore well described by Eq. (1), with the nonlinear term omitted.

We therefore approximate Eq. (1) by:

$$\frac{\partial \psi_s}{\partial t} = A\frac{\partial^2 \psi_s}{\partial z^2} + B|\psi_s|^2\psi_s + g\psi_s \qquad (2)$$

$$\frac{\partial \psi_n}{\partial t} = A\frac{\partial^2 \psi_n}{\partial z^2} + g\psi_n + \Gamma(z, t). \qquad (3)$$

The equations are coupled by the common net gain coefficient g. In the thermodynamic limit the fluctuations in g are small [10], and it is determined by the condition that the total power expectation $\frac{1}{L}\int dz(|\psi_s|^2 + \langle|\psi_n|^2\rangle)$ has a constant value P. We will search for a steady state solution of (2) and (3) consisting of a pulse of power xP, $0 \leq x < 1$, and a continuum background of mean power $(1 - x)P$.

The steady state pulse solution of Eq. (2) with power xP is the well-known chirped soliton pulse [12]:

$$\psi_s(z, t) = [x^2PL/(2L_p)]^{1/2}e^{j\phi t}\text{sech}^{1+j\beta}(xz/L_p), \qquad (4)$$

where the chirp parameter is $\beta = (3/2)[\text{Re}\,AB^*/\text{Im}\,AB^* \pm \sqrt{8/9 + (\text{Re}\,AB^*/\text{Im}\,AB^*)^2}]$, the + (plus) sign for $\mu > \rho$, and − (minus) for $\mu < \rho$, and the pulse width is $L_p = -6|A|^2\beta/(PL\text{Im}\,AB^*)$. The overall net gain g experienced by the pulse is

$$g = -L/(4L_p)\bar{\gamma}_sPx^2,$$
$$\bar{\gamma}_s = (4/3)[\gamma_s - \gamma_g(1 + \beta^2)/(PLL_p)], \qquad (5)$$

$\bar{\gamma}_s$ is the effective nonlinear gain coefficient acting on the pulse; it reduces to γ_s when the soliton condition holds. The dynamical time scale $t_p = (\bar{\gamma}_sP)^{-1}$ is the time scale governing the pulse buildup process [see Eq. (14) below].

The linear Eq. (3) describes the dynamics of noninteracting modes, and is most conveniently studied in Fourier space, where it becomes

$$\dot{a}_k = (g - Ak^2)a_k + \Gamma_k(t), \qquad (6)$$

where a_k and Γ_k are the Fourier transforms of the field ψ and the noise Γ, respectively, and $k = (2\pi/L)n$, n integer. Multiplying the last equation by a_k^* and taking the real part of the expectation value gives, using $\langle\Gamma_k a_k^*\rangle = T$,

$$\dot{x}_k/2 = (g - \gamma_gk^2)x_k + t_c^{-1}, \qquad (7)$$

where $x_k = \langle|a_k|^2\rangle/P$ is the average power fraction in mode k and $t_c = P/T$ is the noise time scale which describes the accumulation of noise power in the cavity. In the steady state $\dot{x}_k$ is zero and we can solve Eqs. (7) for x_k, and find the total noise power fraction

$$1 - x = \sum_k x_k = t_c^{-1}L/(2\sqrt{-\gamma_g g}); \qquad (8)$$

the last equality holds in the thermodynamic limit $L \gg L_c$, where $L_c = \frac{\gamma_g P}{LT}$ [10] is the correlation or coherence length. Eliminating g from the last equality gives:

$$g = -\frac{L^2 T^2}{4\gamma_g P^2 (1-x)^2} = -\frac{L}{4L_c} t_c^{-1} \frac{1}{(1-x)^2}. \quad (9)$$

As previously pointed out, in a statistically steady state the pulse and continuum must experience the same gain. Therefore, by using Eqs. (5) and (9), we obtain:

$$\frac{1}{2} x(1-x) = \sqrt{\frac{L_p}{4L_c} \frac{T}{\bar{\gamma}_s P^2}} = \frac{1}{\sqrt{\gamma_L \gamma_t}}, \quad (10)$$

where $\gamma_L = 4L_c/L_p$ and $\gamma_t = t_c/t_p$. Then,

$$x = (1/2)(1 \pm \sqrt{1 - 8/\sqrt{\gamma_L \gamma_t}}), \quad (11)$$

where $\gamma_L \gamma_t = (\gamma_s P^2/T)^2 f_1/f_2$, $f_1 = 2(\mu - \rho)^2 \times [\sqrt{8(\mu-\rho)^2 + 9(1+\mu\rho)^2} - 8\rho(\mu-\rho) - 9(1+\mu\rho)]$, $f_2 = 9(1+\rho^2)^2[3(1+\mu\rho) - \sqrt{8(\mu-\rho)^2 + 9(1+\mu\rho)^2}]$. Therefore, a necessary condition for mode locking is obtained by requiring that the roots x are real:

$$\sqrt{\gamma_L \gamma_t} \geq 8. \quad (12)$$

When Eq. (12) does not hold, gain balance is impossible to satisfy, and the only stationary configuration is cw. For the soliton condition, $\gamma_L = \gamma_t = \gamma_s P^2/T$, and Eqs. (10)–(12) conform with established results regarding the existence of a metastable pulsed state in this case [10].

An important consequence of Eqs. (10) and (12) is that the pulse power at threshold is nonzero. In fact, it is always *precisely* one half of the intracavity power, regardless of the values of other system parameters. In particular, it follows that pulse formation is abrupt, signifying a first order phase transition, as previously observed under the soliton condition [3,10]. It can also be shown that mode locking is statistically more stable than cw when $\sqrt{\gamma_L \gamma_T} \geq 9$, that corresponds to pulse-to-total power ratio greater than two thirds.

The destabilization threshold can be obtained from Eq. (12) (with the equal sign):

$$[T/(\gamma_s P^2)]_{\text{threshold}} = (1/8)\sqrt{f_1/f_2} \equiv f(\mu, \rho). \quad (13)$$

f gives the maximal allowable noise for a given pumping or the minimal mode-locking power for a given noise level, as shown graphically by color coding in Fig. 1.

There are two notable properties of the function f. First there is a region in parameter space (μ, ρ) where $f^2 \leq 0$ or has a nonzero imaginary part—the very dark (green) color in Fig. 1. This is a region where, because of refractive effects, the overall net gain acting on the pulse is *positive*. Mode locking is not possible under such conditions [2] because positive overall net gain inevitably leads to noise buildup and an ultimate cw state, an observation which is confirmed and generalized by our analysis. Second, the maximal value of f is not achieved under soliton conditions $\mu = \rho$ (where $f = 1/8$). For large μ and ρ, it occurs for $\mu = 2\rho$ where $f \to 1/(4\sqrt{3})$.

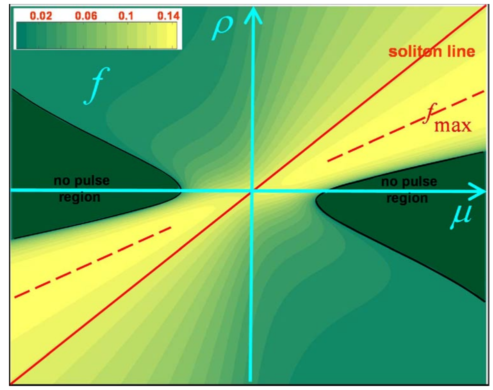

FIG. 1 (color online). The normalized pulse destabilization noise power $f(\mu, \rho)$ [Eq. (13)] shown as a color coded function. Bright (yellow) colors correspond to high noise thresholds (high resistance to noise), and dark (green) colors to low thresholds. At the very dark regions pulse formation is impossible (zero threshold). Note that the region for the most immune pulses against noise destabilization (maximum f) is not the soliton condition line $\rho = \mu$, but asymptotically at $\rho = \mu/2$.

Now we address the question of stability of the pulse states. It follows from Eq. (11) that above threshold there are two pulse states, one with pulse power above and one with pulse power below one half of the intracavity power. Under the soliton conditions, the two states correspond to the extrema of the free energy function—the state with the high pulse power to a minimum, and the lower pulse power to a maximum, from which it follows that the former is (locally) stable while the latter is unstable with respect to small perturbation. The same picture is shown below to hold in the general case.

Our analysis relies on the assumption that the perturbed pulse waveform retains the chirped soliton shape [Eq. (4)] with perturbed pulse power. This is tantamount to the *dynamical* stability of the chirped pulse solution of the *noiseless* master equation, which has been shown to hold in the entire range of system parameters with negative net gain, provided that gain is saturated enough [8]. Then Eq. (2) yields the following equation for the pulse power fraction $x(t)$

$$\dot{x}/2 = (L/4L_p)\bar{\gamma}_s P x^3 + gx; \quad (14)$$

this equation has to be combined with Eqs. (7) for the continuum modes powers, and g is then determined by the condition $x + \sum_k x_k = 1$.

For the stability analysis we let $x = x^{(0)} + x^{(1)}$, $x_k = x_k^{(0)} + x_k^{(1)}$, $g = -g^{(0)} - g^{(1)}$, where $x^{(0)}$ is a solution of (10), and $x_k^{(0)}$ and $-g^{(0)}$ are the steady state values, while $x^{(1)}$, $x_k^{(1)}$, and $g^{(1)}$ are assumed to be small. The linearized equations are

$$\dot{x}^{(1)}/2 = 2g^{(0)}x^{(1)} - g^{(1)}x^{(0)}, \quad (15)$$

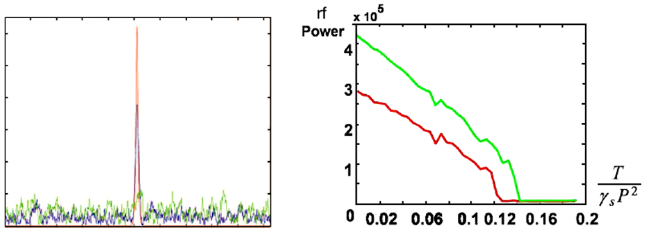

FIG. 2 (color online). Left: simulations of the waveforms in the cavity with $\mu = \rho = 1$ and noise powers that are zero (red), below (blue), and above (green) the threshold of the pulse destabilization level. Right: numerical rf power vs normalized noise level $T/\gamma_s P^2$; we used $\rho = \mu = 1$ (red), and $\rho = 1$, $\mu = 2$ (green).

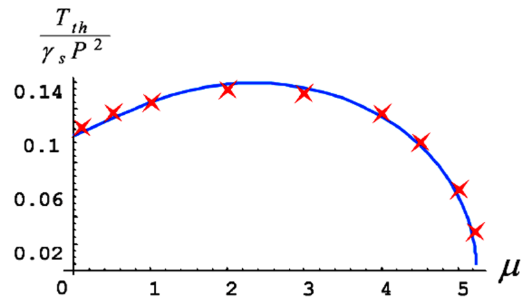

FIG. 3 (color online). Analytical (line) and numerical (stars) threshold values of $T/\gamma_s P^2$ vs μ, for $\rho = 1$. Note again that the best stability against noise is achieved here at $\mu = 2.353$, rather than at $\mu = 1$, where the soliton condition holds.

$$\dot{x}_k^{(1)}/2 = (-g^{(0)} - \gamma_g k^2)x_k^{(1)} - g^{(1)}x_k^{(0)}. \qquad (16)$$

For eigensolutions with eigenvalue $\lambda/2$, Eq. (16) gives:

$$x_k^{(1)} = x_k^{(0)} g^{(1)}/(\lambda + g^{(0)} + \gamma_g k^2). \qquad (17)$$

The linear stability of the steady state depends on whether the system Eqs. (15) and (16) has eigensolution with positive λ. For such an eigensolution Eqs. (17) can be summed over k giving

$$x^{(1)} = (x^{(0)} - 1)g^{(1)}/[\sqrt{g^{(0)}(g^{(0)} + \lambda)} + (g^{(0)} + \lambda)] \quad (18)$$

using also Eq. (9). The last equation is combined with Eq. (15) to yield a homogeneous equation for $x^{(1)}$, which has a nonzero solution if and only if

$$x^{(0)} = (2 - s)/(3 + \sqrt{1 + s}), \qquad s \equiv \lambda/g^{(0)}. \qquad (19)$$

s is the nondimensionalized version of the eigenvalue λ, which is related by the last equation to the steady state pulse power fraction $x^{(0)}$. In the domain of validity, $-1 < s < 2$, $x^{(0)}$ is a decreasing function of s, and $x^{(0)}_{(s=0)} = 1/2$.

Recall that the sign of s determines the stability of the pulsed state. Hence a mode-locked state is stable whenever the pulse power is larger than one half of the intracavity power, marginally stable when it is exactly one half of the intracavity power, and unstable when the pulse is weaker. Note that this stability property is valid, again, *regardless* of the values of the other parameters.

Combining the last result with the analysis above, we observe that solutions given in Eq. (11) above threshold comprise of one stable branch of pulse states with pulse power fraction greater than one half, and one unstable branch with pulse power fraction below one half. The two branches cross at the threshold values of system parameters, where pulsed solutions cease to exist, and below threshold the only statistical steady state is cw.

As an independent test of the theory we performed direct numerical analysis of the dynamical equation (1). Graphical results are shown in Figs. 2 and 3. The left part of Fig. 2 shows three snapshots of the waveform inside the cavity, with zero, subthreshold, and superthreshold noise powers. The qualitative properties conform with the theory. The pulse destabilization noise power T_{th} itself was identified as a point of discontinuity in the rf power $= \int |\psi|^4 dz$, which drops by a large factor of the order of L/L_P; see the right part of Fig. 2.

In this manner T_{th} was measured for several values of ρ and μ. The simulation results agree very well with the theoretical predictions, as seen, for example, in Fig. 3 for the normalized threshold noise power $T_{\text{th}}/\gamma_s P^2$ (stars) and the theoretical prediction (continuous line), for $\rho = 1$ and several values of μ.

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