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Light-mode locking: a new class of solvable statistical physics systems

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Abstract. Passively mode-locked lasers are extended one-dimensional dynamical systems subject to noise, with a nonlinear instability and a global power constraint. We use the recent understanding of the importance of entropy in these systems to study mode locking thermodynamically. We show that this class of problems is solvable by a mean field-like theory, where the nonlinear pulse free energy and entropic continuum free energy compete on the available power, and calculate explicitly the pulse power and mode locking, which occurs when the dimensionless scaled interaction strength $\gamma = 9$. A transfer matrix calculation shows that the mean field theory is exact in the thermodynamic limit, where the number of active laser modes tends to infinity.

Contents

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1. Introduction

Mode-locked lasers, which produce pulse trains of coherent light are an essential experimental tool [1, 2], and serve at the same time as a fascinating paradigm of an extended nonlinear dynamical system [3, 4]. The method of choice for generating ultrashort pulses is passive mode locking, obtained by the action of a saturable absorber, where pulses as short as a few femtoseconds, i.e., 1–2 light cycles, can be formed [5]. As such they have been the subject of many theoretical works. Since the early works of the 1970s [6] mode-locking dynamics is understood in terms of the 'master equation', which describes the temporal evolution of the optical electric field envelope ψ.

However, it has been recently realized [7]–[10] that the purely dynamical approach to mode locking is incomplete, and a full description of the physics of mode locking must take account of the presence of noise in a statistical theory as a fundamental parameter. A clear demonstration of this fact is the experimental observation that a threshold power is needed to reach mode locking, a fact which is outside the scope of the dynamical theory. The effect of cavity noise has been often considered before, e.g., in [11], in a perturbative manner. However, the stabilization of the continuous wave (cw) configuration by noise is an entropic effect, which cannot be captured in perturbation theory, and passive mode locking is a first-order phase transition.

The necessity of analysing noise nonperturbatively has led to the development of statistical lightmode dynamics (SLD). The master equations of SLD, such as equation (1) below, are random nonlinear partial differential equations. In this paper, we show that this equation is solvable by a mean field-like theory in the thermodynamic limit, where the number of active modes tends to infinity. The model is a prototype of a new class of solvable statistical physics systems with rich thermodynamic behaviour. The unique feature of these systems is a destabilizing nonlinearity, which saturates due to a global power constraint. The global constraint is an essential element of SLD, necessary to ensure the existence of a well-defined steady state, in contrast with other statistical physics systems with a global constraint [12]. Mode locking is a thermodynamic phase transition to a configuration, where a macroscopic fraction of the available power is concentrated in a microscopically narrow interval. This property, which is a consequence of the inherent instability in the system, is the basic reason for the solvability of the SLD models, and also the reason that nontrivial thermodynamics is exhibited by these one-dimensional systems.

Our previous theoretical studies of SLD [7, 9] used a simple case of the dynamical master equation, assuming a band-limited waveform instead of a parabolic spectral filtering appropriate for most laser systems. In [9], the exact mean field theory was developed and applied to the simplified model, which may be viewed as a heuristic approximation for the more physical SLD model, which is solved in this paper.

The aforementioned master equation is presented in section 2, and includes, in addition to the saturable absorber and noise, the spectral filtering of the gain, group velocity dispersion and Kerr nonlinearity. We study the equation under the soliton condition, an often-assumed relation between the equation coefficients, where the invariant measure is a Gibbs distribution [7]. It is shown that there are two natural independent length scales in the system, the pulse width and the correlation length, and that the thermodynamic limit is reached when the cavity length is much larger than one of the microscopic length scales. It follows that the thermodynamics is determined solely by the dimensionless parameter γ, four times the ratio of the correlation length to the pulse width. An important corollary of this result is a scaling relation: the intracavity power needed to maintain mode locking grows as the inverse of the cavity length, when length is decreased while other physical parameters are held fixed. In section 3, the mean field theory of SLD is developed and applied to an exact and explicit calculation of the free energy. Using the expression for the free energy, we calculate all thermodynamic quantities of interest, including the pulse power as a function of γ, and the thermodynamic phase diagram, which consists of one ordered, mode-locked phase for large γ, and one disordered, non-mode-locked phase for small γ. Coexistence occurs precisely at $\gamma = 9$, which is the threshold value beyond which passive mode locking may be spontaneously achieved. As a first-order phase transition, passive mode locking is accompanied by metastable states, and the metastable lifetimes are exponentially large in the number of degrees of freedom [13]. For this reason, the instability threshold is as important as the coexistence point. It is shown that the mode-locked state becomes unstable when $\gamma \leqslant 8$. The cw state, on the other hand, is always metastable, which goes a step towards resolving the self-starting problem [14]. The questions of metastable state lifetimes and self-starting dynamics are beyond the scope of this paper, and are dealt with in forthcoming publications [15]. Finally, in section 4, the mean field arguments are substantiated by a rigorous transfer matrix calculation, which affirms the expression for the free energy derived in section 3.

2. The statistical steady state of passively mode-locked lasers

The temporal evolution of the complex envelope of the electric field $\psi(x, t)$ in a cavity of length L is governed by the master equation [7, 17],

$$\partial_t \psi = (\gamma_g + \mathrm{i}\gamma_d)\partial_x^2 \psi + (\gamma_s + \mathrm{i}\gamma_k)|\psi|^2\psi + g\psi + \eta. \tag{1}$$

The master equation is usually formulated [20] in terms of two time scales, a short scale describing the temportal waveform of ψ and a long scale describing the evolution of the waveform, at a reference point in the cavity between consecutive passes. This approach has complicated boundary conditions involving the two time variables. The traditional applications of the master equation concentrated on the pulse waveform, which occupies only a small fraction of the cavity, and the boundary conditions could therefore be sidestepped. In SLD, on the other hand, we need to consider the full waveform, which includes both the pulse and continuum background. The present formulation overcomes this difficulty by sampling, as in [6], the waveform every roundtrip time, recording its spatial rather than its temporal form. In this formulation, simple periodic boundary conditions in space can be imposed. Since the derivation of the master equation relies on the assumption that ψ does not significantly change during a cavity roundtrip time or its fraction, the spatial waveform is related to the temporal one by a factor of the group velocity v_g at their argument. The real constants $\gamma_g > 0$, γ_d, $\gamma_s > 0$, γ_k, characterizing spectral filtering, group velocity dispersion, saturable absorption, and Kerr nonlinearity, respectively, as defined through equation (1) are straightforwardly obtained from the usual definitions [17] modified by appropriate factors of the roundtrip time and v_g.

Noise of spontaneous emission and other sources is modelled in equation (1) by the random term η, which is a (complex) Gaussian process with covariance $\langle \eta^*(x, t)\eta(x', t')\rangle = 2TL\delta(x - x')\delta(t - t')$. The constant T, which plays the role of temperature, is the rate of injection of power into the laser by the noise. Throughout most of this paper T is viewed as a free parameter, which may be controlled experimentally by coupling an external noise source. The slow saturable gain g, as shown in [9], may be chosen without significant loss of generality, such that it sets the total intracavity power $\|\psi\|^2 = \frac{1}{L}\int_0^L \mathrm{d}z|\psi(z)|^2$ to a fixed value P. g becomes then a Lagrange multiplier for the fixed power constraint.

In this paper, we consider equation (1) in the special but important case that $\gamma_s + \mathrm{i}\gamma_k$ is a real multiple of $\gamma_g + \mathrm{i}\gamma_d$, known as the soliton condition. While in practice it is not always easy to obtain, it is often a good working approximation, and facilitates the derivation of an explicit expression for the invariant measure. When the soliton condition does not hold, the invariant measure is not known and a different theoretical approach is needed, which is beyond the scope of the present work [16].

When the soliton condition holds, one can define a 'Hamiltonian' functional,

$$H[\psi] = \int_0^L \mathrm{d}z \left(-\tfrac{1}{2}\gamma_s|\psi(z)|^4 + \gamma_g|\psi'(z)|^2\right), \tag{2}$$

such that, as shown in [10], the invariant measure $\rho[\psi]$ of equation (1) is

$$\rho[\psi] = Z^{-1}\mathrm{e}^{-H[\psi]/(LT)}\delta(P - \|\psi\|^2), \tag{3}$$

$$Z = \int [\mathrm{d}\psi][\mathrm{d}\psi^*]\mathrm{e}^{-H[\psi]/(LT)}\delta(P - \|\psi\|^2). \tag{4}$$

Note that the power constraint is enforced explicitly. The study of steady-state properties of equation (1) is now reduced, as in [7, 9], to that of an equilibrium statistical mechanics system with partition function Z. It should be stressed that H is not actually the Hamiltonian of the system, and does not have dimensions of energy, but rather of power × distance/time. We also point out that, under the assumed soliton condition, H and therefore the statistical steady state, do not depend on the strength of the refractive coefficients γ_k and γ_d [10]; the latter affect only the dynamics.

The functional H is superficially quite similar to the critical Ginzburg–Landau (GL) functional, the paradigm for the effective description of continuous phase transition [18]. In equation (2), however, in contrast with the GL functional, the coefficient of the quartic term is negative, making the constant (cw) configuration unstable and H unbounded from below. In contrast with standard models of phase transitions, the instability is met by the global constraint of fixed total power rather than by a local term of higher degree. Consequently, configurations where a macroscopic fraction of the total power is concentrated in a narrow pulse become statistically significant, and the resulting thermodynamics is radically different from the GL thermodynamics. This feature, which is a natural property of SLD, is the reason the one-dimensional system exhibits an ordering transition, which is impossible in the one-dimensional GL model, and why the mean field-like arguments of section 3 are valid.

The physical content in equations (2)–(4) is of a competition between the ordering nonlinearity and disordering noise. Each physical effect can be used to construct an associated natural length scale: the pulse width

$$L_p = \frac{4\gamma_g}{\gamma_s PL} \tag{5}$$

is inversely proportional to the saturable absorption strength, while the correlation length

$$L_c = \frac{\gamma_g P}{LT} \tag{6}$$

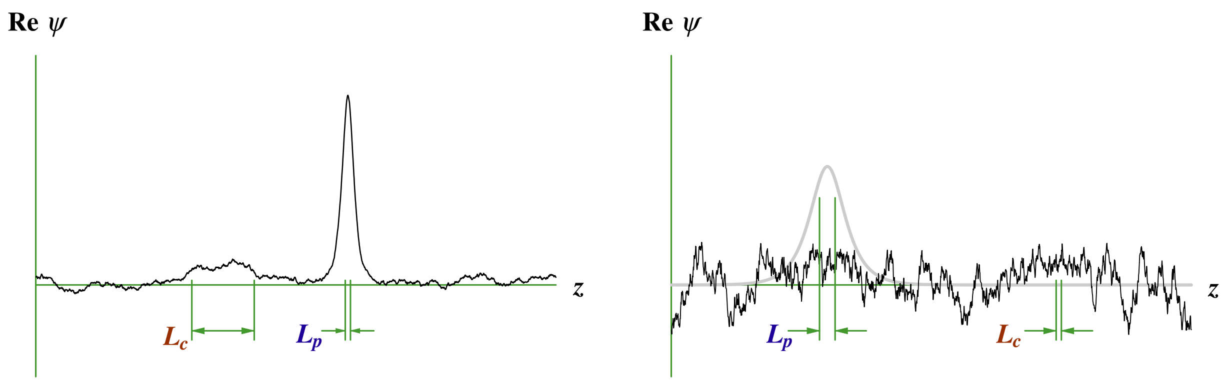

Figure 1. A typical realization of the real part of the envelope ψ of the electric field in the laser cavity, in the mode-locked phase (left), where the correlation length is larger than the pulse width, and in the cw phase (right), where the pulse width is larger than the correlation length.

is inversely proportional to the noise power injection rate. Both length scales are properly thought of as microscopic, as they are much smaller than L, the cavity length. $N_{\rm ac} \equiv L/\min(L_p, L_c)$ is the number of active modes, which can be a large number in multimode lasers. In SLD, $N_{\rm ac}$ is the number of participating degrees of freedom, so $N_{\rm ac} \to \infty$ is the thermodynamic limit, which we study in this paper, neglecting terms of $O(1/N_{\rm ac})$. Thermodynamic quantities, and in particular the mode-locking threshold must then be completely determined by the sole dimensionless parameter

$$\gamma = \frac{4L_c}{L_p} = \frac{\gamma_s P^2}{T}. \tag{7}$$

γ is the normalized interaction strength; $\mathcal{T} = 1/\gamma$ is the normalized temperature.

When $L_p \ll L_c$, γ is large, the nonlinearity dominates, and the equilibrium is an ordered, mode-locked phase, where the power P is divided between a single-pulse and continuum fluctuations, as in the left panel of figure 1; when $L_c \ll L_p$, γ is small, the noise dominates, and the equilibrium is a disordered phase, where the electric field consists only of spatially homogeneous fluctuations, as in the right panel of figure 1. The analysis presented below shows that these two phases are connected by a first-order phase transition, with coexistence for $\gamma = 9$. Metastable pulsed configurations exist for $\gamma > 8$, while cw is metastable for all γ. In temperature terms, cw is the high-temperature phase which coexists with the low-temperature mode-locked phase at $\mathcal{T} = 1/9$, and 'superheated' mode locking may persist up to $\mathcal{T} = 1/8$. The free energy, pulse power and RF power have similar simple expressions, see equations (17) and (18), and figure 2.

SLD can also be formulated in Fourier space [7, 9, 10]. Then the degrees of freedom are the discrete Fourier modes ψ_k, $k = 2\pi n/L$ for integer n, and H takes the form

$$H = \sum_k \gamma_g k^2 |\psi_k|^2 - \gamma_s \sum_{k+k'=m+m'} \psi_k \psi_{k'} \psi_m^* \psi_{m'}^*. \tag{8}$$

The mode phasors can be pictured as complex 'spin' variables, which interact nonlocally via the ordering quartic saturable absorber term in H, and the spectral filtering acts as a damping term which increases quadratically with the displacement of the mode from the band centre. Mode ordering is present when the mode variables acquire nonzero expectation values $\langle\psi_k\rangle$, with k-independent phases, similar to spin ordering in ferromagnetism. Since each mode interacts with all others, one can make the mean field approximation, replacing the ψ_k variables by their expectation values in the interaction term in H, with negligible error in the thermodynamic limit [9]. However, due to the presence of the quadratic filtering term, $\langle\psi_k\rangle$ depends on k, making the Fourier space calculation cumbersome. Therefore, real space analysis is pursued in the following.

The fact that the parameter γ defined in equation (7) determines mode locking has an important practical implication. In most lasers, a main source of white noise is spontaneous emission whose power injection rate dictates a minimal 'temperature' [11]

$$T_{\mathrm{se}} = \frac{\mu(G-1)\hbar\omega}{t_R^2}, \tag{9}$$

where G is the gain factor, μ the population inversion factor, ω is the optical carrier frequency, and t_R is the roundtrip time. Therefore, assuming that γ_s and P are kept fixed, γ is proportional to the square of the roundtrip time, when the repetition rate of the laser is varied, and for fixed γ_s the mode-locking power is inversely proportional to t_R. Furthermore, the nonlinearity strength in some common mode locking methods employed in ultrafast optics, such as the nonlinear polarization rotation technique [19], is proportional to the cavity length, and there the dependence of γ and the threshold power on t_R is even stronger. Therefore, spontaneous emission can place a limit on the pulse repetition rate obtainable by passive mode locking.

Our analysis of the statistical mechanics problem equations (3) and (4) follows the textbook approach of calculating the free energy $F = -T\log Z$, from which other thermodynamic quantities follow. However, the functional integral in equation (4) is not well-defined in the continuum limit. Mathematically this is not a serious problem, since the invariant measure is well-defined [21], but in order to use Z and F, we need to give a precise meaning to the functional integral, as the continuum limit of a regularized version, where the integration is finite-dimensional. Given a regularization scheme with N ψ integration, we define the regularized partition function

$$Z_N = a_N \int \prod_{n=1}^{N} \left(\tfrac{\gamma_g}{L^2 T}\mathrm{d}\psi_n \mathrm{d}\psi_n^*\right) \mathrm{e}^{-\frac{H_N[\psi]}{LT}} \delta(P - \|\psi\|_N^2), \tag{10}$$

where H_N and $\|\cdot\|_N$ are regularized versions of H and $\|\cdot\|$. The integration measure is multiplied by factors of γ_g/L^2T to make Z_N dimensionless, and by a_N, a regularization scheme- and N-dependent dimensionless constant, to make $Z = \lim_{N\to\infty} Z_N$ finite. The limit is independent of the regularization scheme up to an unimportant multiplicative constant.

3. Mean field calculation of the free energy

As a preliminary step towards the calculation of F, we examine the problem in two limits. In the first, $T \to 0$, ρ is dominated by configurations, which minimize H for a given total power P. These configurations, which are stationary solutions to equation (1) with the random term η droppes, are the well-known soliton-like pulses

$$\psi_0(x) = \mathrm{e}^{\mathrm{i}\phi}\sqrt{\frac{PL}{2L_p}}\,\mathrm{sech}\left(\frac{z-z_0}{L_p}\right), \tag{11}$$

which depend on two real parameters, the pulse position $0 \leqslant z_0 < L$ and phase $0 \leqslant \phi < 2\pi$.[2] The $T \to 0$ limit of F, the pulse free energy f_p, is proportional to the minimal value of H

$$f_p = \frac{H[\psi_0]}{L} = -\frac{\gamma_s^2 L^2 P^3}{48\gamma_g} = -\frac{\gamma T}{12}\frac{L}{L_p}. \tag{12}$$

The second solvable limit is $\gamma_s \to 0$, where H becomes quadratic and the partition function Z_c can be calculated explicitly, by diagonalizing the H in the Fourier representation and using the Fourier decomposition of the delta function $\delta(x) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}w}{2\pi\mathrm{i}}\mathrm{e}^{zw}$. The Gaussian integration gives (referring to equation (10)),

$$Z_c = \lim_{N\to\infty} a_N \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\mathrm{e}^{Pz}\prod_{n=-N}^{N}\pi((2\pi n)^2 + z)^{-1}. \tag{13}$$

a_N may be chosen such that

$$Z_c = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\mathrm{e}^{z}\prod_{n=-\infty}^{\infty}\left(1 + \frac{Lz}{L_c(2\pi n)^2}\right)^{-1}. \tag{14}$$

When $L \gg L_c$, the Euler–Maclaurin sum formula for the infinite product yields

$$\prod_{n=-\infty}^{\infty}\left(1 + \frac{Lz}{L_c(2\pi n)^2}\right)^{-1} = \exp\left(-\sqrt{\frac{Lz}{L_c}}\right), \tag{15}$$

and then the contour integral in equation (14) can be evaluated using the saddle point method; the resulting expression for $\gamma_s \to 0$ limit of F is the continuum free energy

$$f_c = \frac{L^2T^2}{4\gamma_g P} = \frac{LT}{4L_c}. \tag{16}$$

Although the preceding expressions equations (12) and (16) for the free energy were obtained in limiting cases, we now argue that the pulse free energy f_p and the continuum free energy f_c may be combined into an expression for F valid for every γ. The argument is based on the following premise: configurations ψ which contribute significantly to Z are such that $\psi(z) = \mathrm{O}(1)$ for most z, with possibly a few narrow intervals of total width $\mathrm{O}(L_p)$, where $\psi(z) = \mathrm{O}(\sqrt{L/L_p})$. Let w, $0 \leqslant w \leqslant P$ be the total power concentrated in regions where ψ is large. For z values where ψ is small, the nonlinear term in H is negligible, and the existence of regions of large ψ affects the statistics of the small ψ region only in that the total available power for fluctuations is $P - w$

[2] Pulses of the form (11) obey the periodic boundary conditions only approximately. However, since the discrepancy is exponentially small in the large parameter L/L_p, it is completely negligible.

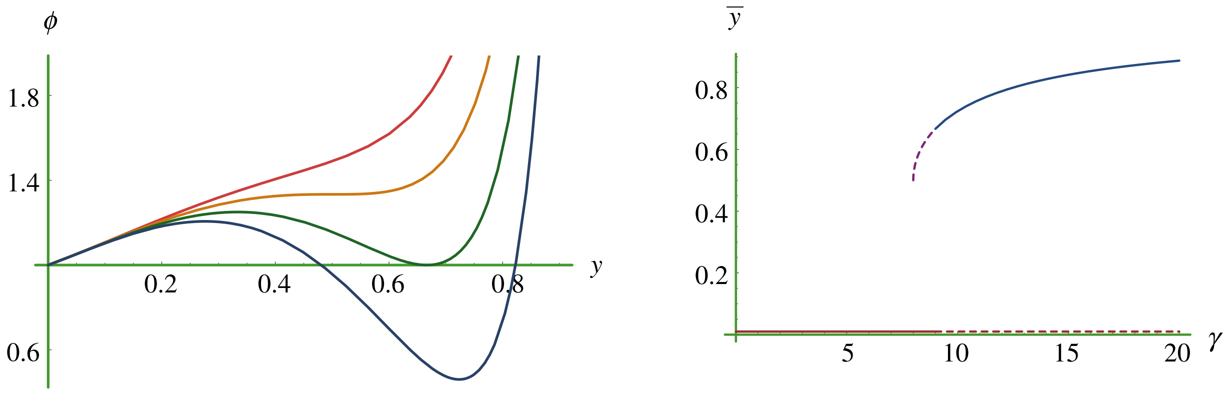

Figure 2. Left: the dimensionless free energy ϕ as a function of the dimensionless pulse power y is shown for $\gamma = 7$–10 with higher values of γ corresponding to lower curves and colder colours. Curves with two (local) minima correspond to systems with a metastable state. Coexistence occurs at $\gamma = 9$, where the values of ϕ at the two minima are equal. Right: the dimensionless pulse power $\bar{y}$ as a function of the dimensionless interaction strength γ. $\bar{y} = 0$ corresponds to cw. Values shown in broken lines correspond to metastable states.

rather than P. The regions of small ψ therefore contribute $f_c|_{P\to P-w}$ to the total free energy. Similarly, the regions of large ψ are so narrow that noise-induced fluctuations make negligible contribution to the free energy in them, so that the large ψ regions contribute $H[\psi]/(LT)$ to F. By the principle that F is minimized by the Gibbs distribution, the waveform in the large ψ regions should be such that $H[\psi]$ is minimized, that is, ψ will assume a soliton-like shape with total power w, and contribute $f_p|_{P\to w}$ to F. We claim that since the mean field-like argument presented here holds when L_p/L is small, it becomes exact in the thermodynamic limit, where $L_p/L \to 0$. This claim is established in a controlled calculation in section 4 below.

We can define a pulse-power dependent free energy $f(w) = f_c(w) + f_p(P - w)$, which is the analogue of the Landau function in mean field theory [22]. As in mean field theory, $F = \min_w f$, and the minimizer $\bar{w}$ is the pulse power, which plays the role of the order parameter: $\bar{w} = 0$ means that the thermodynamically stable phase is cw. It is convenient to express the thermodynamics using the dimensionless variables, $\gamma = \gamma_s P^2/T$, $y = w/P$ and $\phi(y) = (4L_c/TL) f(w)$; the scaled free energy ϕ is

$$\phi(y) = -\frac{\gamma^2 y^3}{12} + \frac{1}{(1-y)}. \tag{17}$$

The function ϕ has the following properties (see figure 2): For $\gamma \leq 8$ $\phi(y)$ has a single minimum at $y_0 = 0$, and for $\gamma > 8$, there is a second (local) minimum at $y_1(\gamma) = \frac{1}{2}(1 + \sqrt{1 - 8/\gamma})$, which becomes a global minimum, when $\gamma \geqslant 9$ in the standard scenario of first-order phase transition, see figure 2. Hence

$$\bar{y} = \begin{cases} 0 & \gamma < 9 \\ \frac{1}{2}(1 + \sqrt{1 - 8/\gamma}) & \gamma > 9, \end{cases} \tag{18}$$

with coexistence at $\gamma = 9$. In terms of mode-locking equation (18) means that when T is large enough or γ_s or P are small enough that $\gamma < 9$, the thermodynamically stable state of the system is cw, and mode locking is stable when $\gamma > 9$.

However, equation (18) may fail to predict the observed behaviour of an actual system, which can reside for a long time in a metstable state analogous to supercooling or superheating. It follows from the properties of ϕ that metastable mode locking may persist for $\gamma > 8$, and that cw is metastable for any positive γ. The latter result is a very robust property of the system, stemming from the fact that f_c is linear in w near $w = 0$, while f_p, which is intrinsically nonlinear, vanishes faster as $w \to 0$. The metastability of cw is the starting point for the SLD study of self-starting [15].

It is also straightforward to calculate other thermodynamic quantities, such as $M \equiv \langle \int \mathrm{d}z |\psi|^4 \rangle$, which is directly proportional to the experimentally measurable RF power [8]. In the thermodynamic limit, M receives its dominant contribution from the pulse, whence

$$M = \frac{L^2}{3L_p}\bar{w}^2. \tag{19}$$

4. Transfer matrix calculation of the free energy

In this section, we reconsider the problem of calculating the partition function. Instead of evaluating the integral expression (4), we solve the equivalent transfer matrix equation (26). The solution is compatible with mean field theory, establishes rigorously the thermodynamic results of the previous section such as equations (17) and (18), and demonstrates our earlier claim that mean field theory is exact in the thermodynamic limit.

The transfer matrix method is better suited for fixed boundary condition problems. Accordingly, we use the fixed boundary condition partition function $\tilde{Z}$ and free energy $\tilde{F} = -\log \tilde{Z}$ with endpoint values $\psi(0) = \psi_{\mathrm{i}}$, $\psi(L) = \psi_{\mathrm{f}}$. In this calculation, it is more convenient to use system parameters that do not depend on the system size, so we define

$$\tilde{Z}(\psi_{\mathrm{i}}, \psi_{\mathrm{f}}, \tilde{P}, L) = \int_{\psi(0)=\psi_{\mathrm{i}}}^{\psi(L)=\psi_{\mathrm{f}}} [\mathrm{d}\psi][\mathrm{d}\psi^*] \times \mathrm{e}^{\int_0^L \mathrm{d}z\left(\frac{1}{2}\alpha_1|\psi(z)|^4 - \alpha_2|\psi'(z)|^2\right)} \delta\left(\int_0^L \mathrm{d}z|\psi(z)|^2 - \tilde{P}\right). \tag{20}$$

The mean field arguments of the previous section are straightforwardly adapted to the fixed boundary conditions. As before, we define a free energy function $\tilde{f}(w)$ of the pulse power w, which is the sum of the pulse free energy $\tilde{f}_p(w)$ and continuum free energy $\tilde{f}_c(w)$. The continuum free energy is a bulk quantity, and insensitive to boundary conditions in the thermodynamic limit; it is therefore given by equation (16), which in the present parametrization reads

$$\tilde{f}_c = \frac{L^2}{4\alpha_2(\tilde{P} - w)}. \tag{21}$$

Like f_p, $\tilde{f}_p$ is obtained by minimizing the functional H in the space of functions of total power w. However, the fixed boundary conditions break the translation invariance, and the minimization is achieved, when pulses are created near the boundaries. Standard variational methods yield the following result: the minimizer of H takes one of four possible forms, which include a partial pulse at each boundary. The forms differ in the relative position of pulse maximum and the boundary point. The four possible values of $\tilde{f}_p$ are distinguished by the four possible sign choices in

$$\tilde{f}_p = \frac{2\sqrt{\alpha_2}}{3\alpha_1}\left(2\lambda^{\frac{3}{2}} \pm \left(\lambda - \frac{\alpha_1}{2}|\psi_{\rm i}|^2\right)^{\frac{3}{2}} \mp \left(\lambda - \frac{\alpha_1}{2}|\psi_{\rm f}|^2\right)^{\frac{3}{2}}\right) - \lambda w, \tag{22}$$

where in the first sign choice the upper (lower) sign refers to the possibility that a pulse maximum lies at $z > 0\,(z < 0)$. The second sign choice refers similarly to the position of the pulse maximum relative to the $z = L$ boundary. λ is a Lagrange multiplier for the power constraint, given implicitly by

$$w = \frac{2\sqrt{\alpha_2}}{\alpha_1}\left(2\sqrt{\lambda} \pm \sqrt{\lambda - \tfrac{1}{2}\alpha_1|\psi_{\rm i}|^2} \mp \sqrt{\lambda - \tfrac{1}{2}\alpha_1|\psi_{\rm f}|^2}\right). \tag{23}$$

It is next to be shown that the free energy $\tilde{F} = \min(\tilde{f}_p(w) + \tilde{f}_c(\tilde{P} - w))$ solves the transfer matrix equation (27) which is derived from equation (26) for $\tilde{Z}$. Like Z, $\tilde{Z}$ needs to be defined using a limiting procedure with a properly scaled functional measure (see equation (10)). As explained above, results do not depend on the regularization procedure. Here it is convenient to use lattice regularization, rather than the regularization scheme employed in equation (14), approximating derivatives with finite differences and integrals with Riemann sums. Let $\tilde{Z}_\delta$ denote the lattice regularization of $\tilde{Z}$ with a small lattice spacing δ. It satisfies the identity

$$\tilde{Z}_\delta(\psi_{\rm i}, \psi_{\rm f}, \tilde{P}, L) = \int \frac{\alpha_2}{\delta\pi}\,{\rm d}\psi\,{\rm d}\psi^*\,\tilde{Z}_\delta(\psi_{\rm i}, \psi, \tilde{P} - \delta|\psi|^2, L - \delta)\exp\left(\frac{\delta}{2}\alpha_1|\psi|^4 - \frac{\alpha_2}{\delta}|\psi_{\rm f} - \psi|^2\right). \tag{24}$$

Taylor expansion of the right-hand side in powers of δ and $\psi - \psi_{\rm f}$ gives

$$\begin{aligned}\tilde{Z}_\delta(\psi_{\rm i}, \psi_{\rm f}, \tilde{P}, L) = {\rm Re}\int {\rm d}^2\psi \exp\left(-\frac{\alpha_2}{\delta}|\psi_{\rm f} - \psi|^2\right)\Big(1 + 2(\psi - \psi_{\rm f})\partial_{\psi_{\rm f}} + (\psi - \psi_f)^2\partial_{\psi_{\rm f}^2} \\ +|\psi - \psi_f|^2\partial_{\psi_{\rm f}}\partial_{\psi_{\rm f}^*} - \delta|\psi_{\rm f}|^2\partial_{\tilde{P}} - \delta\partial_L + \tfrac{1}{2}\delta\alpha_1|\psi_{\rm f}|^4\Big)\tilde{Z}_\delta + {\rm o}(\delta),\end{aligned} \tag{25}$$

where the arguments of $\tilde{Z}$ are now the same on both sides of the equation. The ψ integration in (25) is Gaussian, and gives in the continuum limit $\delta \to 0$ the transfer matrix equation for $\tilde{Z}$

$$\frac{1}{\alpha_2}\partial_{\psi_{\rm f}}\partial_{\psi_{\rm f}^*}\tilde{Z} - |\psi_{\rm f}|^2\partial_{\tilde{P}}\tilde{Z} - \partial_L\tilde{Z} + \tfrac{1}{2}\alpha_1|\psi_{\rm f}|^4\tilde{Z} = 0. \tag{26}$$

The equation for $\tilde{F}$ is obtained by substituting $\tilde{Z} = {\rm e}^{-\tilde{F}}$ in (26) and keeping only leading terms in L/L_p,

$$\frac{1}{\alpha_2}|\partial_{\psi_{\rm f}}\tilde{F}|^2 - |\psi_{\rm f}|^2\partial_{\tilde{P}}\tilde{F} + \tfrac{1}{2}\alpha_1|\psi_{\rm f}|^4 = 0. \tag{27}$$

The task of showing that $\tilde{F}$ satisfies the differential equation (27) will be accomplished by showing that the free energy obtained by minimizing $\tilde{f}$ with respect to w for each of the four possible sign choices in equation (21) satisfies equation (27). Since $\tilde{F}$ is equal to one of these functions, it also solves this equation. To this end, we recall the definition of $\bar{w}$, the minimizer of the free energy, and define $\bar{\lambda}$, which is related to $\bar{w}$ via equation (23). It is straightforward to obtain an explicit expression for $\bar{\lambda}$

$$\bar{\lambda} = \frac{L^2}{4\alpha_2(\tilde{P} - \bar{w})^2}. \tag{28}$$

The partial derivatives of $\tilde{F}$ then read

$$\partial_{\tilde{P}} \tilde{F} = \bar{\lambda}, \qquad \partial_{\psi_{\mathrm{f}}} \tilde{F} = \mp \psi_{\mathrm{f}}^* \sqrt{\alpha_2(\bar{\lambda} - \tfrac{1}{2}\alpha_1 |\psi_{\mathrm{f}}|^2)}, \tag{29}$$

where the upper or lower sign choice follow to the second sign choice in equations (22) and (23). Equation (27) is identically satisfied by these explicit expressions for both sign choices, and the demonstration is complete.

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