

## Formation and Annihilation of Laser Light Pulse Quanta in a Thermodynamic-like Pathway

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We present a theoretical and experimental study of multiple pulse formation in passively mode-locked lasers. Following a statistical-mechanics approach, the study yields a thermodynamic-like “phase diagram” with boundaries representing cascaded first order phase transitions. They correspond to abrupt creation or annihilation of pulses and a quantized rf power behavior, as system parameters (noise and/or pumping levels) are varied, in excellent accordance with the experiments. Remarkably, individual pulses carry an almost constant quantum of energy.

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Lasers operating in a pulse regime can develop more than one pulse running in their cavities [1]. That is, at any given instant, the optical energy is sharply concentrated around several points along the cavity rather than at one point (a single pulse regime) or evenly distributed over the cavity (continuous wave operation). This multipulse regime can be triggered, for example, by a suitable form of absorptive nonlinearity. These lasers exhibit rich and complicated dynamics, the modeling of which [2–6] requires inclusion of nonlinearities of higher order than the common quartic (Landau-Ginzburg-like) term. A remarkable feature often exhibited by these lasers is the quantization of the pulse energies [7,8]. That is, all pulses possess nearly fixed energy. In addition, they interact and can form ordered bunch structures [9].

Recently it has been shown [10] that the interplay between laser nonlinearities and noise, such as the inevitable spontaneous emission noise, can account for formation and destruction of a single pulse in passively mode-locked (PML) lasers. This discontinuous formation and destruction of pulses was identified as a first order phase transition, governed by the balance between interaction (due to the nonlinearity) and entropy (associated with noise). The study is taken beyond the simple *mechanics* view of pulses and modes in lasers, to a *statistical-mechanics* picture, where a central role is played by basic thermodynamic quantities such as “temperature” (noise), entropy, and free energy.

In this Letter we present new theoretical and experimental results concerning the thermodynamics of pulses in lasers. We focus on the very creation and annihilation of pulses in a multipulse regime as the pumping and the intracavity noise level are varied. We find that as noise is increased and/or pumping is decreased, pulses successively disappear, one by one, in what is shown to be a cascade of first order phase transitions. This novel thermodynamic multiphase light system gives rise to a quantized rf power effect. The theory relies on an equilibrium-like statistical-mechanics model that is solved analytically. The qualitative and quantitative agreement between theory and experiment is excellent.

The dynamics of the (classical) electric field in a PML laser is commonly described by the complex Ginzburg-Landau equation, which can also include high order nonlinearities [3], especially in the context of multiple pulse operation. In our study we wish to include the effect of noise as well, which would render the above description difficult to handle analytically or numerically. We therefore construct here a much simpler model of the dynamics, which is capable, as the experiment demonstrates, of capturing the key features of noise-dependent formation and destruction of pulses.

We divide the cavity to  $N$  equal intervals, such that  $N/\tau$  ( $\tau$  is the cavity round-trip time) is of the order of the bandwidth of the laser, which means that the duration of a pulse is of the order  $\tau/N$ . Then we write an equation of motion for the energy  $x_m$  at the  $m$ -th interval:

$$\frac{dx_m}{dt} = s(x_m)x_m + gx_m + \sqrt{x_m}\Gamma_m(t). \quad (1)$$

$s(x)$  is the effective nonlinear gain for the energy in the interval,  $t$  is the long scale time variable over which the laser evolves between round-trips,  $g$  is the overall net gain (originating from the slow amplifier and effective linear losses),  $\Gamma_m$  is real white Gaussian noise satisfying  $\langle \Gamma_m(t)\Gamma_m(t') \rangle = T\delta_{mn}\delta(t-t')$ , and the last term is in the Stratonovich interpretation [11]. Properly modifying  $g$  one can always set  $s(0) = 0$ .

When the  $m$ -th interval is occupied by a pulse, an equation of the form of Eq. (1) has been established in previous studies and both  $s(x)$  and the form of the noise term given in Eq. (1) were obtained: The latter for solitons [12], and the former also for other types of pulses [5]. If on the other hand the  $m$ -th interval belongs to the continuum, its energy is small enough such that nonlinearities are negligible. Then neglecting  $s(x_m)$  and the dependence of  $g$  on  $x_m$  in Eq. (1), we are left with the same equation of motion that we would obtain for the energy of a sample of a band-limited function under the effect of white Gaussian noise.

We assume that the interaction between different intervals is weak enough such that the equations of motion of  $x_m$ 's are not coupled, apart from a global constraint of

constant total energy  $\mathcal{P}$ , introduced by choosing  $g$  as the appropriate Lagrange multiplier [10,13].

While in previous energy rate equation studies [5] the energy of the continuum was represented by a single degree of freedom, in our study this is not appropriate. When noise is present *the continuum carries essentially all the entropy* [13] and hence it is crucial that it is represented by many degrees of freedom.

Following the same steps as in Refs. [10,13,14], one can show that the invariant measure for Eq. (1) is a Gibbs distribution described by the partition function

$$Z_N(T, \mathcal{P}) = \int dx e^{\frac{1}{T} \sum_{m=1}^N S(x_m)} \delta\left(\sum_{m=1}^N x_m - \mathcal{P}\right) \quad (2)$$

where  $S(x) = \int_0^x s(x') dx'$  and  $dx$  denotes integration with respect to all  $x$ 's from zero to infinity.

From Eq. (2) it is clear that the  $x_m$ 's are bounded by  $\mathcal{P}$ . If  $s(x)$  is an increasing function of  $x$  for  $x < \mathcal{P}$ , which can happen when a saturable absorber is present and  $\mathcal{P}$  is small enough, pulses usually do not split. Pulse splitting occurs typically when  $s(x)$  is an increasing function at  $0 < x < x_s$  for some  $x_s$  and is a *decreasing* function for  $x > x_s$  [15–17], which is the situation, for example, in additive pulse mode locking [15]: It is intuitively clear that one pulse with an energy much higher than  $x_s$  will be less favorable than many pulses with energies of the order  $x_s$ . We henceforth assume the above described structure of  $s(x)$ , and show that when  $\mathcal{P} \gg x_s$ , Eq. (2) predicts formation of a variable number of pulse “quanta”, i. e., pulses with nearly constant energy [16]. Pulse quanta are spontaneously and abruptly created and annihilated when  $\mathcal{P}$  and  $T$  are varied.

The model studied in Ref. [13] is a special case of (2), in which the function  $S$  is quadratic. There, however, the model was derived under somewhat different conditions than in the present work. Thus, if  $s$  is chosen linear at the origin ( $S$  is quadratic), the present results recapture those of Ref. [13] for small  $\mathcal{P}$ , but we expect that they do not describe the actual behavior of the multipulse laser at small powers.

In the thermodynamic theory of our system, the formation of a single pulse is a first order phase transition [10,13,14], and the formation of multiple pulse configurations is a first order transition between different ordered “phases”, reminiscent of structural phase transitions in solids [18]. To show the remarkable resemblance to the thermodynamic phase picture with the predictive power of the statistical-mechanics theory, we refer right at the beginning to Fig. 1. The latter shows the theoretical phase diagram derived below from Eq. (2), and also experimental measurements as described below. Each phase is labeled by the number of pulses.

We proceed to outline the analysis of the statistical-mechanics model of Eq. (2), which is similar to the one given in detail in Ref. [13]. We make the physically appropriate assumption that  $N \gg 1$ , i. e., that  $\tau$  is much

longer than the duration of a single pulse (or equivalently the number of modes, which is of the order  $N$ , is large). This amounts to taking the thermodynamic limit in Eq. (2).

We start by assuming a specific asymptotic form (as  $N \rightarrow \infty$ ) for the partition function  $Z_N^{(m)}$  of configurations with  $m$  pulses or less

$$Z_N^{(m)}(T, \mathcal{P}) = A_N^{(m)}(T, \mathcal{P}) e^{-N[F_m(T, \mathcal{P})-1]}, \quad (3)$$

where  $A_N^{(m)}(T, \mathcal{P})$  is subexponential in  $N$  and  $F_m(T, \mathcal{P})$ , the free energy, is the global minimum of

$$f_m(x_1, \dots, x_m) = -\frac{1}{NT} \sum_{j=1}^m S(x_j) - \ln\left(\mathcal{P} - \sum_{j=1}^m x_j\right) \quad (4)$$

for nonnegative  $x$  values such that  $\sum x_j \leq \mathcal{P}$ . Let  $n$  be the smallest  $m$  for which the minimum of  $F_m(T, \mathcal{P})$  with respect to  $m$  is attained, and let  $X_1, \dots, X_n$  be the corresponding minimizer. Our statement then is that  $Z_N$  approaches  $Z_N^{(n)}$  in the thermodynamic limit, and in particular  $n$  is the number of the pulses per round-trip and  $X_1, \dots, X_n$  are their energies. If  $n$  and the  $X_j$ 's are to have a finite thermodynamic limit,  $T$  has to scale like  $1/N$ , so that  $\mathcal{T} \equiv NT$ , the total power of noise, has a finite limit. The necessity of rendering the parameters of the system  $N$  dependent is discussed in [13].

Equations (3) and (4), follow from Eq. (2) if one assumes that as  $N \rightarrow \infty$  all but a finite number  $m$  of the  $x$  variables are  $O(1/N)$ . This is justified by a rigorously controlled approximation in  $1/N$  based on a recursive equation for  $Z_N$ , as in Ref. [13]. The upshot is that the task of calculating thermodynamic properties is reduced to that of finding the minima of  $f$  for different values of  $\mathcal{P}$  and  $\mathcal{T}$ . In this manner, not only the number of pulses per round-trip is obtained, but many other quantities of interest, such as the order parameter

$$Q = \sum_{j=1}^N \langle x_j^2 \rangle = \sum_{j=1}^n X_j^2, \quad (5)$$

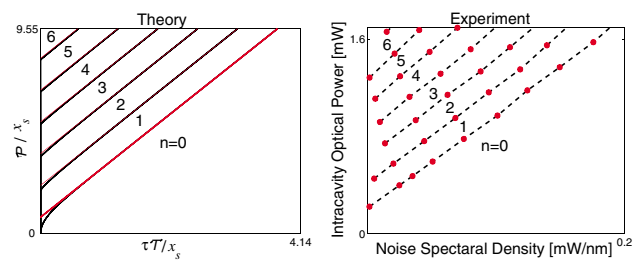


FIG. 1 (color online). Experimental (right) and theoretical (left) phase diagrams. The theoretical graph shows the number of pulses as a function of the intracavity energy  $\mathcal{P}$  and the total noise power  $\mathcal{T}$ , assuming Eq. (12). The curves of discontinuity have the thermodynamic-like meaning of first order phase transitions. The straight lines are asymptotes of the transition lines for  $\mathcal{P} \gg x_s$  [Eq. (11)].

which is proportional to the experimentally measurable rf power of the photocurrent [19].

Since  $m$  is finite, finding the minimum of  $f_m$  for a specific choice of  $s(x)$  is straightforward, although in general it includes a numerical solution of a set of transcendental equations. However for the structure of  $s(x)$  we consider in this Letter, asymptotic expressions for  $\mathcal{P} \gg x_s$ , which is the domain of the multipulse regime, can be obtained. Standard minimization techniques show then that the minima of  $f_m$  are usually obtained at two types of configurations: the first has  $n \leq m$  nonzero  $x$  values all having the same value  $X$ , i. e., configurations of  $n$  equipotent pulses; and the second, which is much rarer, is  $n - 1$  values of  $x = X$  and one additional value  $x = X' < X$ . A sufficient condition for excluding the second type of solution is that  $s(x)$  increases at least as fast as it decreases:

$$s(x_1) = s(x_2), x_1 < x_2 \Rightarrow |s'(x_1)| \geq |s'(x_2)|. \quad (6)$$

Assuming the first type of solution, the problem reduces to the minimization of the function

$$f(n, x) = -\frac{nS(x)}{\mathcal{T}} - \ln(\mathcal{P} - nx), \quad (7)$$

with respect to two variables,  $n$  (integer) and  $x$  ( $0 \leq x \leq \mathcal{P}/n$ ). The minimizer  $X$  satisfies

$$s(X) = \frac{\mathcal{T}}{\mathcal{P} - nX}. \quad (8)$$

Clearly  $X$  is the common pulse energy and the order parameter is  $Q = nX^2$ .

The asymptotic regime relevant to a multipulse operation,  $\mathcal{P} \gg x_s$ , is most readily analyzed by considering the minimization of the same function  $f$  appearing in Eq. (7) but with  $n$  replaced by a real valued variable  $\nu$ . Minimizing with respect to  $x$  and  $\nu$  together immediately gives  $X = X_*$ , where  $X_*$  is the solution of

$$S(X_*) = X_*s(X_*). \quad (9)$$

$X_*$  is the quantized pulse energy. From the assumptions made above on  $s(x)$  it follows that  $X_*$  is unique as long as it exists, and  $X_* > x_s$  (remarkably enough). We use the notation  $s_* = s(X_*)$ ,  $S_* = S(X_*)$ . Then writing  $X = X_* + \delta$  and  $\nu = n + \{\nu\}$  with  $n$  integer and  $|\{\nu\}| \leq \frac{1}{2}$ , and putting back in Eq. (8) gives

$$\frac{\delta}{X_*} = \frac{\{\nu\}s_*}{s'(X_*)(\mathcal{P} - \nu X_*) - ns_*}, \quad (10)$$

if  $\delta \ll X_*$ . Since  $s'(X_*) < 0$  while  $s_* > 0$ , indeed  $\delta \ll X_*$  provided either  $\mathcal{P} - nX \gg X_*$  or  $n \gg 1$ . At least one of them is certain to hold whenever  $\mathcal{P} \gg X_*$ . This asymptotic region corresponds to the upper part of Fig. 1, and is characterized by a nearly quantized pulse energy.

The phase diagram in the quantized pulse regime is very simple: The transition from an  $(n - 1)$ -pulse configuration to an  $n$ -pulse configuration occurs approximately when  $\nu$  is half an odd integer. Using Eq. (8)

once more gives the transition temperature  $\mathcal{T}_n(\mathcal{P})$

$$\mathcal{T}_n(\mathcal{P}) = s_*\mathcal{P} - (n - 1/2)S_*, \quad (11)$$

i.e., the transition curves are approximately equally spaced straight lines, as seen in Fig. 1.

In order to illustrate our results we chose

$$s(x) = \tau^{-1} \sin\left(\frac{\pi x}{2x_s}\right). \quad (12)$$

The motivation of this choice is the sinusoidal transmissivity in additive pulse-mode locking [15], and although  $s(x)$ , the net gain experienced by a pulse, somewhat differs from the bare transmissivity [5], we disregard this difference here, since the multiple pulse regime is anyway governed by the neighborhood of  $X_*$ . Results for Eq. (12) are shown in Fig. 1 for the number of pulses and in Figs. 2 and 3 for the order parameter  $Q$ .

Our experimental study was conducted on a setup similar to a recently reported one [19], where the first order phase transition associated with the formation and destruction of a single pulse was lucidly demonstrated. It consists of a fiber ring laser with PML by nonlinear polarization rotation technique [8,15], with amplified spontaneous emission noise injected from an external source, in order to have direct control over the spectral power of the additive noise in the cavity. Here we used a shorter laser with a round-trip time of 100 ns, corresponding to approximately 20 m of total cavity length, including a 4.3-m long erbium-doped fiber amplifier with small signal gain of 6 dB/m. By proper adjustment of the polarization controllers (PCs), PML operation was established with generation of sub-ps pulses. As observed in a variety of PML fiber lasers [1,7,8,15], excessive pumping (above the self-starting power threshold) led to the formation of multiple pulses per round-trip that in general were randomly distributed over the cavity. However, for certain positions of PCs, stable bunches of nearly identical pulses were formed, with approximately constant spacing between adjacent pulses (that ranged from a few to hundreds of ps, depending on PCs positions [20]).

As the noise or pumping levels were varied, three regimes of responses of the pulse bunch were observed,

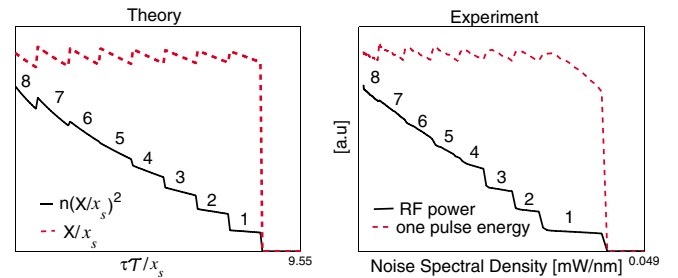


FIG. 2 (color online). Experimental (right) and theoretical [left, with Eq. (12)] plots of the order parameter  $Q$  (solid line) and the mean pulse energy (dashed line) as functions of the noise spectral power, for a fixed intracavity energy.

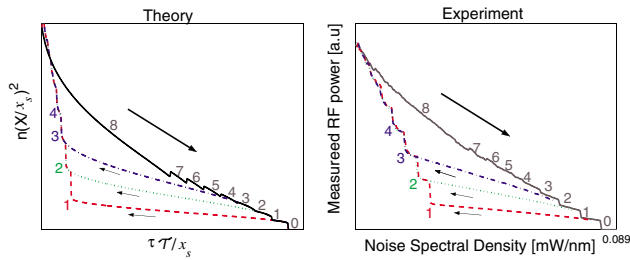


FIG. 3 (color online). Experimental (right) plot of the order parameter  $Q$  as a function of increasing (solid line) and decreasing (broken lines) noise spectral power, demonstrating three different hysteresis paths, with excellent agreement to the theoretical plot (left), with the choice  $s(x) = a_1x - a_3x^3 + a_5x^5$ . For a motion picture demonstration of the experimental results, see Ref. [20].

depending on the position of the PCs: Variations in the spacing between adjacent pulses (contraction and expansion) with constant pulse energies, variations in the pulse energy with constant spacings, and variation in both spacing and energy (most common regime).

Figure 1 shows the experimental phase diagram measured as follows: for several pumping powers the injected noise level was raised gradually from zero, the pulses disappeared one by one, and the transition temperatures and average output optical powers were recorded. Such a behavior was previously observed as the pumping power was decreased [8]. The experimental results in Fig. 1 were obtained at a small varying energy and spacing regime, but the structure of the phase diagram was common to all the regimes mentioned above. Figure 1 demonstrates good agreement between theory and experiment.

Theoretical and experimental plots of the order parameter  $Q$  [Eq. (5)], and the energy per pulse as a function of the injected noise level for a fixed pumping power are shown in Fig. 2. Experimentally they were obtained at the regime with nearly constant pulse spacing and noticeable pulse energy variations, by measuring the laser output with a fast photodiode and an rf power meter [19] or a sampling oscilloscope (all having 50 GHz bandwidth), correspondingly. The pulse energy is nearly constant, with deviations of about 5%. These deviations are well described by the theory.

Figure 3 shows additional theoretical and experimental plots of the order parameter dependence on noise for both increasing and decreasing of the noise level. Typically to first order phase transitions, the system exhibited hysteresis [20]. The hysteresis can be theoretically estimated using an analog to the Arrhenius formula. Since the lifetime of a metastable phase is exponential in the height of the barrier surrounding the corresponding local minimum of  $f$ , it can be much longer than the time over which the system parameters are varied.

We conclude by noting how powerful the combination of statistical mechanics and laser physics can be, leading

us to a new view and findings that can be significant to both fields.

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- [20] See EPAPS Document No. E-PRLTAO-93-010439 for a motion picture of the experimental results shown in Fig. 3 including pulse formation and annihilation. Typical pulse width is 500 fs, and the pulse bunch length is about 1/300 of the cavity length. A direct link to this document may be found in the online article's HTML reference section. The document may also be reached via the EPAPS homepage (<http://www.aip.org/pubservs/epaps.html>) or from <ftp.aip.org> in the directory /epaps/. See the EPAPS homepage for more information.