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Phase transition theory of pulse formation in passively mode-locked lasers with dispersion and Kerr nonlinearity

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Abstract

We extend our statistical mechanics approach (Phys. Rev. Lett. 89 (2002) 103901) to passively mode-locked lasers to also include dispersion and the Kerr nonlinearity and in particular the soliton laser case. The theory predicts for these general cases a discontinuous first-order phase transition at the onset of pulses that explains the power threshold behavior of mode-locked lasers.

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1. Introduction

In the course of the recent three decades, much progress has been made in the study of passive mode locking (PML) of lasers. The analytical part of the studies has led to a rather consensual theory [2,3]. There are many nuances of this theory, but in most of the contemporary analytical descriptions of PML the core remains the same: an equation of motion (the master equation), reminiscent to the complex Landau–Ginzburg equation, for the evolution of optical pulses, modeling the action of gain dispersion, group velocity dispersion (GVD), a usually fast saturable absorber, the Kerr nonlinearity and slow saturable gain. This family of theories has been used for extensive study of PML: the pulse shape [4], stability issues [5], the threshold behavior [6], noise characteristics [7] of the pulses and many more. A recent review is given in [3].

Most of the existing theories of the PML threshold behavior [6,8–11] consider noise or some sort of incoherence to be responsible for it. When the effect of noise on PML is studied through the Langevin master equation, noise is often assumed to be small, which facilitates linearization of the Langevin master equation [7]. This way, the lowest moments describing the statistics of the output signal of PML lasers can be calculated. As we find below, crucial features of the system can be missed in this procedure when the noise is large.

In a recent work [1] we presented a statistical mechanics approach for a many interacting mode laser system. We have shown that such systems

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exhibit phase transition of the first kind, a result of the interplay between entropy and (a generalization of) energy in a system of many interacting modes, where the role of temperature is taken by noise. Contrary to small noise in the laser, which causes noise in its output, increasing the noise (or equivalently decreasing the signal) to a certain critical value causes an abrupt breakdown of the pulses and a transition to a noisy CW operation. The pulse formation in lasers is shown to be similar to melting–freezing behavior in solid–liquid and para-ferromagnet transitions. We have recently experimentally demonstrated the predicted first-order phase transition in a passively mode-locked fiber laser [12], showing the thermodynamic melting–freezing path.

In this paper we extend the study of the threshold behavior of PML [1] to include GVD, the Kerr nonlinearity and a more common description of the optical saturable gain. We extend the "Hamiltonian" formalism [1] to include gain saturation. We discuss the purely dissipative motion, the motion induced by the gain and losses (linear and nonlinear) alone, and present a dynamical picture of the balance between forces and noise, reflecting the balance between interaction and entropy. We then write the Fokker–Planck equation associated with the Langevin equation of motion and look for steady-state solutions which are functions of the Hamiltonians alone. We find such solutions when the soliton condition is satisfied [3], which extends our theory from [1] to this important case. Beyond the soliton condition we investigate the Langevin equations of motion numerically, for general values of dispersion and the Kerr nonlinearity. We find that the equations consistently exhibit the first-order phase transition behavior. We suggest this mechanism as an explanation to the passive mode-locking threshold behavior.

2. "Hamiltonian" formulation of the master equation

The common equation of motion [3] of PML (the "master equation"), which is usually written in the time domain, can be written in the spectral domain as follows:

$$\dot{a}_m = (g_m - l_m + \mathrm{i}\phi_m)a_m + (\mathrm{i}\gamma_k - \gamma_\mathrm{s}) \times \sum_{j-k+l-m=0} a_j a_\mathrm{k}^* a_l + \Gamma_m. \tag{1}$$

The boundaries of the cavity admit only discrete axial modes into the laser, the slowly varying amplitudes of which are denoted by a_m. $\dot{a}_m$ stands for $\mathrm{d}a_m/\mathrm{d}\tau$, where τ is a dimensionless variable measuring time in units of the cavity roundtrip time. g_m denotes the net gain at the wavelength corresponding to m and is signal dependent. ϕ_m represents the phase acquired by the mth mode after a roundtrip, which in particular describes dispersion. l_m represents the wavelength-dependent linear loss. γ_s and γ_k represent the nonlinear self-amplitude (saturable absorber) and self-phase modulation (Kerr nonlinearity) coefficients, and Γ_m is a white Gaussian Langevin additive noise term which in particular describes quantum noise [7]. We assume the Γ's to be Gaussian, white and uncorrelated

$$\begin{aligned} &\left\langle \Gamma_m(t_1)\Gamma_n^*(t_2)\right\rangle = 2T\delta_{mn}\delta(t_1 - t_2), \\ &\langle \Gamma_m(t_1)\Gamma_n(t_2)\rangle = 0 \end{aligned} \tag{2}$$

with $\langle\cdot\rangle$ denoting ensemble averaging. T is the spectral power of the noise, which functions as temperature.

It is well known that lasers and in particular PML lasers are stable due to gain saturation [5]. When the response of the laser amplifier and the evolution of the waveform are much slower than the cavity roundtrip time, the gain at a given moment is determined by the whole waveform all over the cavity. The most common model for gain saturation in that case is

$$g = \frac{g_0}{1 + \mathscr{P}/P_\mathrm{sat}}, \tag{3}$$

where

$$\mathscr{P} = \sum_m a_m a_m^* \tag{4}$$

and P_sat is the saturation power of the amplifier. Eq. (3) describes a wavelength-independent amplifier: the filtering action of the amplifier is embedded in l_m and its signal dependence has been neglected. This common approximation is often satisfying [5], but wavelength-dependent saturation behavior is easily incorporated in our formalism by defining

$$g_m = \frac{g_m^0}{1 + \mathscr{G}/G_{\rm sat}}, \tag{5}$$

where

$$\mathscr{G} = \sum_m g_m^0 a_m a_m^*. \tag{6}$$

Eq. (5) reflects the fact that in a homogeneously broadened amplifier the gain saturation is determined by the overlap between the optical spectrum and the gain profile [13]. $G_{\rm sat}$ is the analog of the saturation power for that case, and can be related to its fundamental properties [13].

Defining

$$\mathscr{H}_{\rm R} = -\sum_m \phi_m a_m a_m^* - \frac{\gamma_k}{2} \sum_{j-k+l-m=0} a_j a_k^* a_l a_m^* \tag{7}$$

and

$$\mathscr{H}_{\rm I} = \sum_m l_m a_m a_m^* - \frac{\gamma_k}{2} \sum_{j-k+l-m=0} a_j a_k^* a_l a_m^* - G_{\rm sat} \ln(G_{\rm sat} + \mathscr{G}), \tag{8}$$

it is straight forward to verify that (1) can be rewritten as

$$\dot{a}_m = \frac{\partial(\mathscr{H}_{\rm R} - {\rm i}\mathscr{H}_{\rm I})}{{\rm i}\partial a_m^*} + \Gamma_m. \tag{9}$$

Changing variables to the real and imaginary parts of a ($a_m = a_m^{\rm R} + {\rm i}a_m^{\rm I}$) we have

$$\begin{aligned} \dot{a}_m^{\rm R} &= \frac{\partial \mathscr{H}_{\rm R}}{\partial a_m^{\rm I}} - \frac{\partial \mathscr{H}_{\rm I}}{\partial a_m^{\rm R}} + \Gamma_m^{\rm R}, \\ \dot{a}_m^{\rm I} &= -\frac{\partial \mathscr{H}_{\rm R}}{\partial a_m^{\rm R}} - \frac{\partial \mathscr{H}_{\rm I}}{\partial a_m^{\rm I}} + \Gamma_m^{\rm I}, \end{aligned} \tag{10}$$

$\Gamma_m^{\rm R}$ and $\Gamma_m^{\rm I}$ are the real and imaginary parts of Γ_m. Observing Eq. (10), we see clearly that $\mathscr{H}_{\rm R}$ generates a Hamiltonian motion and $\mathscr{H}_{\rm I}$ generates a purely dissipative motion (a "gradient flow").

Eqs. (10) are Langevin equations describing a random process, which can be equivalently described by the associated Fokker–Planck equation [14]

$$\begin{aligned} \dot{\rho} = &\sum_m \frac{\partial}{\partial a_m^{\rm R}} \left\{ \left(\frac{\partial \mathscr{H}_{\rm R}}{\partial a_m^{\rm I}} - \frac{\partial \mathscr{H}_{\rm I}}{\partial a_m^{\rm R}} \right) \rho \right\} \\ &+ \sum_m \frac{\partial}{\partial a_m^{\rm I}} \left\{ \left(-\frac{\partial \mathscr{H}_{\rm R}}{\partial a_m^{\rm R}} - \frac{\partial \mathscr{H}_{\rm I}}{\partial a_m^{\rm I}} \right) \rho \right\} \\ &+ T \sum_m \left(\frac{\partial^2}{\partial a_m^{\rm R^2}} + \frac{\partial^2}{\partial a_m^{\rm I^2}} \right) \rho, \end{aligned} \tag{11}$$

where ρ is the statistical distribution of the modes a_m and T is half the spectral power of the Langevin force. As shown in [1], when $\mathscr{H}_{\rm R} = 0$ it is straight forward to see that the distribution

$$\rho(a_1, \ldots, a_N) \propto {\rm e}^{-\mathscr{H}_{\rm I}/T} \tag{12}$$

is the exact solution of Eq. (11) with $\dot{\rho} = 0$ (the steady-state solution). The solution (12) has far reaching consequences, since it is identical by structure to the statistical distribution of a canonical ensemble. We elaborate this case in the next section (Section 3). The more general cases that include the Kerr nonlinearity and dispersion ($\mathscr{H}_{\rm R} \neq 0$) are discussed afterwards (in Sections 4 and 5).

3. Purely dissipative "motion" ($\mathscr{H}_{\rm R}$ = 0)

For the case of zero GVD and no Kerr nonlinearity ($\mathscr{H}_{\rm R} = 0$) the dynamics of the system is gradient flow ($\dot{a}_m^{\rm R} = -\partial\mathscr{H}_{\rm I}/\partial a_m^{\rm R}$, $\dot{a}_m^{\rm I} = -\partial\mathscr{H}_{\rm I}/\partial a_m^{\rm I}$). This is useful for additional intuitive understanding of the dynamics induced by the saturable absorber, gain saturation and filtering. Let us examine the evolution of $\mathscr{H}_{\rm I}$ without noise and for $\mathscr{H}_{\rm R} = 0$. We have then

$$\frac{{\rm d}\mathscr{H}_{\rm I}}{{\rm d}\tau} = -\sum_m \left[\left(\frac{\partial \mathscr{H}_{\rm I}}{a_m^{\rm R}} \right)^2 + \left(\frac{\partial \mathscr{H}_{\rm I}}{a_m^{\rm I}} \right)^2 \right], \tag{13}$$

which means that $\mathscr{H}_{\rm I}$ keeps decreasing until a stationary point of $\mathscr{H}_{\rm I}$ is reached. If the stationary point is not a minimum, then a small perturbation will push $\mathscr{H}_{\rm I}$ down the hill again. If it is a local minimum, then the point is locally stable: after a small perturbation the system will return to the minimum, but a large one will throw the system away without necessarily going back. Only a global minimum of $\mathscr{H}_{\rm I}$ will be a truly stable solution to Eq. (1). It follows therefore that a truly stable solution to $\mathscr{H}_{\rm I}$ will exist if and only if $\mathscr{H}_{\rm I}$ is bounded from below. This provides a convenient tool for stability analysis, and a mathematical embodiment to the commonly used intuitive arguments on what lasers "like" and "dislike": lasers tend to minimize $\mathscr{H}_{\rm I}$.

An immediate result of this stability analysis technique is that the master equation admits no globally stable solutions: it is easy to see from Eq. (8) that $\mathscr{H}_{\mathrm{I}}$ is not bounded from below. The reason for this is that in the simplest model of absorption–saturation we used, which takes into account only its the lowest nonlinear term, increasing the optical power enough will eventually turn the absorber into an amplifier, which is physically incorrect. This problem can be fixed by adding to the master equation a "stabilizing" term which will turn the local minimum of the mode-locked configuration into a global one. An example for such a term is the next one in the expansion of the saturable absorber.

When noise is present, along with sliding down, the system will be randomly thrown from state to state. This random motion is likely to draw the system away from "rare" states (like the minimum of $\mathscr{H}_{\mathrm{I}}$), or in other words, noise tends to increase the entropy of the system. As mentioned above, just like in statistical mechanics, this competition between minimizing $\mathscr{H}_{\mathrm{I}}$ and maximizing entropy leads to the Gibbs distribution, with the noise power taking the role of temperature.

Contrary to linearization techniques, which can provide information on the few lowest moments only, Eq. (12) is an *exact* solution for the *entire distribution* $\rho(a_1,\ldots,a_N)\propto \mathrm{e}^{-\mathscr{H}_{\mathrm{I}}/T}$ of the waveforms (or modes) in the laser. Needless to say – equipped with an exact solution for the distribution one can thoroughly study the noise properties of PML lasers.

More interesting is that the threshold behavior of PML is embedded in Eq. (12): order (mode locking) emerges in the laser upon "cooling". This can be shown analytically too, through the mean field approximation [1]. Eq. (12) establishes an intimate relation between PML lasers and equilibrium statistical mechanics and phase transition theory, and indeed it can be shown that the occurrence of mode locking upon reaching threshold is a phase transition of the first kind. The results provided by statistical mechanics, both through numerical simulation (a Monte-Carlo simulation) and the mean field approximation are shown in Fig. 1.

The connection to phase transition theory and statistical mechanics is not accidental. In fact, the situation in the PML laser is very similar to the one in traditional systems exhibiting order–disorder transitions like melting solids, ferromagnets and many more. Just like the latter, the laser is a system of many interacting degrees of freedom – the modes, with an ordering interaction – the nonlinear saturable absorber, and with temperature (which in our case is replaced by the noise variance) encouraging gain of entropy which opposes the ordering interaction. The theory of phase transitions is a mature theory with many well-established powerful results, which can be now directly applied to PML.

One of the interesting consequences of our analysis is that whenever a saturable absorber is present (for zero GVD and no Kerr nonlinearity) CW is not a stable solution of Eq. (1): it is easy to show that a CW solution is a maximum or a saddle point of $\mathscr{H}_{\mathrm{I}}$. As we have seen, what stabilizes a CW solution is noise alone. This reminds other situations in nature, for example the liquid phase, which remains liquid due to entropy considerations, although its energy would be lower had it solidified, and nevertheless it remains liquid.

4. Analytical solution for a case with $\mathscr{H}_{\mathrm{R}}\neq 0$ including the soliton laser case

Let us now proceed to $\mathscr{H}_{\mathrm{R}}\neq 0$, that is, a laser with nonzero dispersion and Kerr coefficient. Inspired by equilibrium statistical mechanics, where a very fundamental statement is that the statistical distribution is a function of the Hamiltonian only, and even more by the solution (12), we can try to seek for a solution to the steady-state distribution which is again a function of the "Hamiltonians" only. We therefore substitute the solution

$$\rho(a_1,\ldots,a_N)\propto f(\mathscr{H}_{\mathrm{R}}(a_1,\ldots,a_N),\mathscr{H}_{\mathrm{I}}(a_1,\ldots,a_N)) \tag{14}$$

into Eq. (11) and equating to zero leads via a lengthy but trivial calculation to the result that Eq. (12) is the only possible solution of the form (14), and it solves Eq. (11) as long as

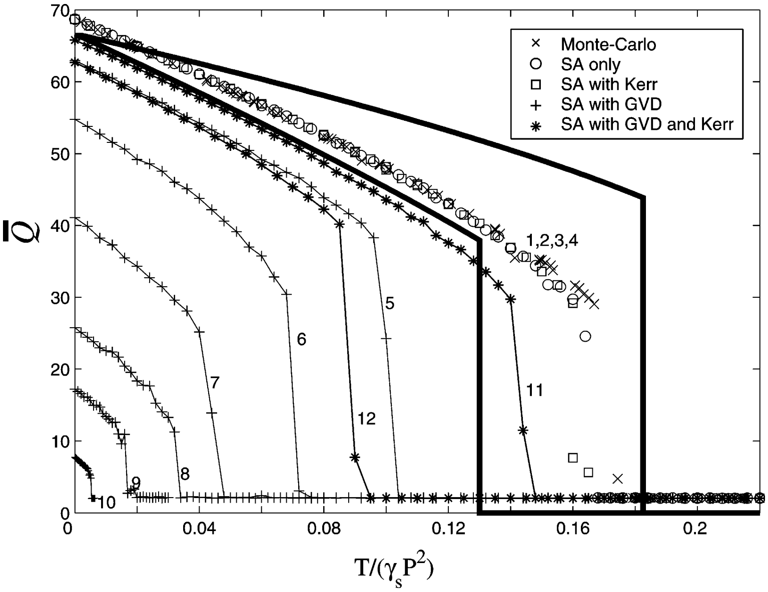

Fig. 1. The order parameter $\bar{\mathcal{Q}}$, inversely proportional to the pulse duty cycle, as function of the generalized temperature T for several choices of the different parameters. Graphs 1–4, which almost coincide, are results of numerical simulations: A Monte-Carlo simulation of Eq. (12) and a direct simulation of the Langevin equation (10), with (graphs 3 and 4) or without (graphs 1 and 2) the Kerr nonlinearity. The thick lines are their analytical mean field based approximations [12]. Graphs 5–10 simulations of Eq. (10) with GVD, and graphs 11 and 12 are the same simulations with both GVD and the Kerr effect. Qualitatively the behavior of graphs 5–12 is very similar to that of graphs 1–4, all exhibiting an abrupt transition to a mode-locked phase. In the simulations we used 100 discrete modes $(a_1, a_2 \ldots, a_{100})$, with equal spectral spacing. The "stabilization" we used is a constraint of constant power $P = \sum |a_k|^2$. T is in units of $\gamma_s P^2$. Graphs 3 and 4 are with $\gamma_k/\gamma_s = 2$ and 3. In graphs 5–12 we took a parabolic dispersion relation for GVD $\phi_m = \gamma_d(n - 50.5)^2$ with $\gamma_d = 0.002, 0.003, 0.004, 0.006, 0.009, 0.02$ for graphs $5, 6, 7, 8, 9, 10$, respectively. In graphs 11 and 12 $\gamma_k = 3\gamma_s$ and γ_d is -0.005 and 0.005, respectively.

$$\sum_m \left(\frac{\partial \mathcal{H}_R}{\partial a_m^I} \frac{\partial \mathcal{H}_I}{\partial a_m^R} - \frac{\partial \mathcal{H}_R}{\partial a_m^R} \frac{\partial \mathcal{H}_I}{\partial a_m^I} \right) = 0, \tag{15}$$

that is, $\mathcal{H}_I$ is a constant of motion under the evolution generated by $\mathcal{H}_R$ alone. Putting the common parabolic profiles for dispersion and gain dispersion and restricting ourselves to Eq. (3) for modeling gain saturation, it is straight forward to show that Eq. (15) holds in particular for *solitons* [3]. The right balance between dispersion and the Kerr nonlinearity is kept.

The conclusion of the above calculation is twofold. First, we have seen that Eq. (12) and hence all the subsequent analysis [1] holds as it is in a soliton laser. Second, we have seen that the steady-state distribution is not a function of the Hamiltonians alone when dispersion and Kerr are nonzero and the soliton condition does not hold. It is therefore an explicit function of the a's, which is difficult to find and presumably also to analyze. We have therefore turned to numerical studies.

5. Numerical study of the general $\mathcal{H}_R \neq 0$ case

The main question we were addressing in the numerical study is whether the sharp breakdown of pulses upon increasing noise occurs with the most general choice of the parameters in Eq. (1). We have therefore conducted tens of numerical experiments, where we numerically solved Eq. (1) and calculated the time average of the normalized order parameter

$$\tilde{Q} = \frac{\sum_{j-k+l-m=0} a_j a_k^* a_l a_m^*}{\left(\sum_m |a_m|^2 \right)^2} \tag{16}$$

as function of the noise level T. This parameter is inversely proportional to the pulsewidth and directly proportional to the number of correlated modes: In the mode-locked phase it is of order of the total number of modes in the band, whereas in the CW phase it is of order one. All the simulations presented in this paper were performed with 100 modes and with the constraint of constant optical power P_0 as explained above. All the numerical experiments exhibited the same sharp breakdown of the pulses, however the critical T depends on the parameters. It seems that pulses with higher chirp parameter are less stable against noise, which is similar to the behavior of stability against small perturbations [3]. Additionally the general effect of the unbalanced GVD and Kerr nonlinearity is to reduce the effective number of locked modes. Some results are presented in Fig. 1.

With nonzero GVD and Kerr nonlinearity, the behavior of a PML laser with noise is qualitatively the same as PML in the purely dissipative case ($\mathscr{H}_R = 0$), and in particular the phase transition occurs, at least for a wide variety of parameters. This is not very surprising, since a phase transition of the first kind is a dramatic singularity, and it is not expected to be removed by slight modifications of the dynamics.

6. Conclusion

We have extended our statistical mechanics approach for many interacting mode systems to general PML lasers that include dispersion and Kerr nonlinearity. Such systems were shown to exhibit a first-order transition. For a specific balance between dispersion and Kerr nonlinearity, that includes the important case of soliton lasers, the proof was given by an exact analysis. For the broader case we used a numerical study.

We also mention again our recent [12] experimental demonstration of the phase transition with a variety of thermodynamic properties of pulse formation in a fiber PML laser. We have shown the melting–freezing transition as the "temperature" noise was varied. The present paper can provide a further understanding of such systems when solitons are incorporated, as often happens in fiber lasers.

We end with an experimental remark. We have analyzed the behavior of PML lasers as the noise power is altered. In an experiment, the mode-locking threshold is usually observed as the power in the cavity is raised. The interaction "energy" responsible for the mode locking is quadratic with the power in the cavity (see the last term in Eq. (8)), while the noise power is close to be linear with it. This explains the existence of a threshold power.

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