Inhibition of modulation instability in lasers by noise

Ariel Gordon and Baruch Fischer

Department of Electrical Engineering, Technion, Haifa 32000, Israel

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It is shown that additive noise can inhibit modulation instability in laser equations of motion. A related self-starting condition for pulsation is obtained by employing a fluctuation–dissipation relation between noise and losses and a statistical mechanics approach. Entropy considerations are shown to play a crucial role. The quantum limit for self-starting is estimated.

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Many equations describing the propagation of light in nonlinear media, either with or without a laser cavity, exhibit modulation instability. Examples are the nonlinear Schrödinger equation,[1] the mode-locking master equation,[2] and the laser Maxwell–Bloch equations.[3]

For modulation instability to manifest itself, sufficient power is required. In a lossy medium, for example, modulation instability must overcome the losses if it is to occur. In the case of the nonlinear Schrödinger equation this condition is conveniently formulated as the requirement that the length scale associated with the nonlinearity be longer than the one associated with the losses.[1] In a laser the situation is different: The gain, as well as a saturable absorber sometimes inserted into the cavity, intensifies the instability.[4] The equations of motion of lasers do not have a stable cw solution when there is a saturable absorber or the Kerr nonlinearity and dispersion are focusing,[5] and yet the existence of such a solution is an experimental fact. It seems that these equations of motion, which explain and predict so much about mode-locked lasers, do not explain how such lasers can operate in a cw regime.

This problem has been addressed in various studies.[3,6–9] Many adhere to the traditional stability analysis approach, trying to find a stable cw solution in different equations.[3,6] For example, when the response of the gain medium is relatively fast, the master equation does have a stable cw solution at low powers.[6] Others go beyond the stability analysis, suggesting different mechanisms that can stabilize a cw operation. Examples are a decoherence process of modes,[7] spurious reflections,[8] and reduction of the laser gain caused by accumulation of random radiation.[9] The stabilizing effect of these mechanisms on a cw regime has been thoroughly investigated.[10–12]

In this Letter we show a fundamental mechanism that can explain cw operation where it is unstable according to the conventional theory: We show that in a laser a cw can be stabilized by noise. This seems counterintuitive, since if a cw is unstable by itself, noise seems only to destabilize it more. However, the physical idea behind stabilization by noise is simple. When noise is present, there must be a net loss per round trip in a steady state, which we refer to as the loss excess. This is a fluctuation–dissipation relation: Since noise constantly supplies additional energy to the system, the only way to keep the energy constant is if the loss is slightly larger than the gain.

When energy uniformly removed from the cavity is returned to it as noise, pulsed solutions are discriminated against. The reason is that pulses, being ordered configurations, have much less volume in phase space, or less entropy in the language of statistical mechanics. The probability of a pulsed solution's being promoted by the noise is therefore extremely small—exponentially small in the number of modes.

The fact that mode locking is an order–disorder phenomenon makes it necessary to take entropy, not only dynamics, into account. This, as well as the fact that mode locking is essentially a many-body problem, makes statistical mechanics very useful in understanding of mode locking.[13,14] Dynamical stability against small perturbations in the initial conditions is not always the correct criterion for distinguishing physical from nonphysical solutions. A dynamically unstable solution can thus be thermodynamically stable, just as the liquid phase of matter is.

In this Letter we show how much noise is required for stabilizing a cw state, which is, in other words, a self-starting condition for a passively mode-locked laser with noise. Following the statistical-mechanics approach, we calculate the depth of the local minimum of free energy representing an unstable cw state and compare it with the noise power ("temperature").

This condition can be estimated from the fluctuation–dissipation relation: If the power of the noise is W and the average intracavity energy is P_0, the rate with which the loss excess removes energy from the cavity is W/P_0. This expression will show up as a loss term in the equation of motion, which will inhibit modulation instability just as ordinary loss does in the case of free propagation. Comparing the loss excess with the nonlinear length scale leads to the same self-starting condition. We compare the result to a previously obtained self-starting condition[7] and estimate the power threshold induced by a fundamental noise source in lasers: spontaneous emission.

We now analyze a general form of a wide class of laser equations of motion and obtain the loss excess term mathematically. Although the idea of loss excess is very general for lasers, being merely a result of energy conservation, to put it as a precise mathematical statement, one must make some assumptions about the equation of motion. Many laser equations of motion are of the form

$$\frac{\mathrm{d}a_m}{\mathrm{d}t} = f_m[a(t)] - \frac{\mathcal{P} - P_0}{\epsilon} a_m(t) + \sqrt{T}\,\tilde{\Gamma}_m(t)\,, \qquad (1)$$

where $a_m(t)$ are the complex slowly varying amplitudes of the axial modes in the laser, t is either the time or a slow time variable that represents the number of round trips the laser has accomplished,[2] and $f_m[a(t)]$ is a function of all the a_m, representing all the deterministic parts of the differential equation except for the gain saturation term. In the case of the mode-locking master equation a_m includes terms representing spectral filtering, group-velocity dispersion, losses, the Kerr nonlinearity, and more.[2] Gain saturation is represented by the second term of Eq. (1), where $\mathcal{P} = \sum_m |a_m|^2$. It represents a slow saturable amplifier that stabilizes the intracavity power $\mathcal{P}$ around a constant value P_0. Such a term can be thought of as a linearization of the formula $g_0/(1 + \mathcal{P}/P_{\mathrm{sat}})$ around the operating power P_0, where g_0 is the small-signal gain, P_{sat} is the saturation power, and $\epsilon > 0$ is the coefficient obtained by the linearization. Regardless, it is a generic term representing a power stabilization mechanism by a slow amplifier. The last term in Eq. (1), where T is the power of the noise per mode, is a complex white Gaussian Langevin force satisfying[15]

$$\langle \tilde{\Gamma}_m(t_1)\tilde{\Gamma}_n^*(t_2)\rangle = 2\delta_{mn}\delta(t_1 - t_2)\,,$$
$$\langle \tilde{\Gamma}_m(t_1)\tilde{\Gamma}_n(t_2)\rangle = 0\,. \qquad (2)$$

We have shown[13,14] that T plays the role of temperature in statistical mechanics: The larger T is, the more states with large entropy are promoted.

If there are N modes, the total power supplied to the laser by the noise is $W = 2NT$. We assume that N is large, approaching infinity, but W remains a finite constant. We also take the limit of $\epsilon \to 0$, which means that we neglect power fluctuations. This specific limiting procedure is needed solely for the loss excess term to clearly decouple and separate from the others and to show up as a constant loss term. Generally, power is a fluctuating quantity and the fluctuation–dissipation relation holds only on average.

The smaller ϵ is, the faster the motion of $\mathcal{P}$ toward P_0. To solve this fast time-scale dynamics first, we perform a scaling of time: $t' = t/\epsilon$. We obtain

$$\frac{\mathrm{d}a_m}{\mathrm{d}t'} = \epsilon f_m[a(\epsilon t')] - (\mathcal{P} - P_0)a_m(\epsilon t') + \left(\frac{\epsilon W}{2N}\right)^{1/2} \tilde{\Gamma}_m(t')\,, \qquad (3)$$

where Eq. (2) holds with t replaced by t'. Multiplying Eq. (3) by a_m^*, taking the real part, and summing over m, we obtain

$$\frac{1}{2}\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t'} = \epsilon\ \mathrm{Re}[a_m^* f_m(a)] - (\mathcal{P} - P_0)\mathcal{P} + \left(\frac{\epsilon W}{2N}\right)^{1/2} \mathrm{Re}(a_m^* \tilde{\Gamma}_m)\,, \qquad (4)$$

where the summation convention for a repeating index is used here and henceforth and the t' dependence has been suppressed. Since the spectral power of the noise term on the right-hand side of Eq. (4) is $\epsilon W P_0/N$, in the limit of $N \to \infty$ it vanishes, and that term can be replaced by its nonvanishing average value[16] $\epsilon W/2$. Since the resulting deterministic equation reaches a steady state much faster than the term $\epsilon\ \mathrm{Re}[a_m^* f_m(a)]$ varies, one can solve for $\mathcal{P}$:

$$\mathcal{P} = P_0 + \frac{\epsilon}{P_0}\left\{\frac{W}{2} + \mathrm{Re}[a_m^* f_m(a)]\right\} + O(\epsilon^2)\,. \qquad (5)$$

Equation (5) can be substituted back into Eq. (1), and in the limit $\epsilon \to 0$ we obtain

$$\frac{\mathrm{d}a_m}{\mathrm{d}t} = f_m(a) - \frac{\mathrm{Re}[a_k^* f_k(a)]}{P_0} a_m + \sqrt{T}\,\tilde{\Gamma}_m - \frac{W}{2P_0} a_m\,. \qquad (6)$$

The second term in Eq. (6) projects out of f_m the component responsible for variations in $\mathcal{P}$. If the last two terms were absent, $\mathcal{P}$ would be a constant of motion of Eq. (6), provided that its initial value were P_0. The last term, which is a constant loss term, is the loss excess. It removes energy from the laser at a rate of W/P_0.

We now present an example: the mode-locking master equation, which is the nonlinear Schrödinger equation with gain, saturable absorption, and possibly other terms. Without gain in this example modulation instability does not occur when the length scale of losses is shorter than that of the nonlinearity.[1] In a laser, replacing the ordinary loss by the loss excess, we obtain the following self-starting condition:

$$\gamma P_0^2 > W\,, \qquad (7)$$

where γ is the nonlinear coefficient of the medium, either the self-amplitude or the self-phase-modulation coefficient. Typical of statistical mechanics, this condition is a comparison between the interaction and the noise ("temperature").

Nonlinear equations with noise are intricate, and, to go beyond the estimation given in relation (7), a precise analysis of the interplay between noise and the nonlinearity, temperature and interaction in the language of statistical mechanics, is needed. As we explained, a physically (thermodynamically) stable state would be a minimum of the free energy of the system. To demonstrate this powerful tool, we derive a self-starting condition for the special case of a saturable absorber alone. We assume that there are N modes equally supported by the laser amplifier and define the order parameter

$$M = \frac{1}{\sqrt{NP_0}}\left|\sum_m a_m\right|\,.$$

Clearly, if the amplitudes and phases of the modes are all equal, that is, a locked state, M approaches 1. Whenever there is a large number of modes that are in phase, M is finite, whereas in a cw phase M is $O(1/\sqrt{N})$, approaching zero. In Refs. 13 and 14 we studied the free energy of this system and its two minima, one representing the cw and the other, pulsation. When the levels of these minima meet, a phase transition occurs. That is, if the power is lower (higher) than at that point, the cw (pulsation) is the globally stable state.

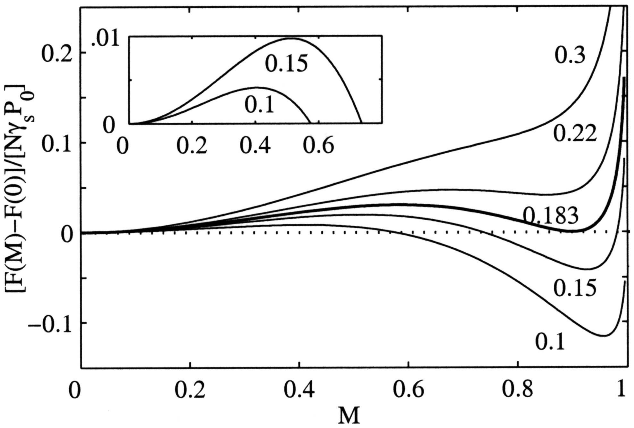

Fig. 1. Plot of the free energy, obtained from the mean field theory,[13,14] as a function of the order parameter M for different values of T. The number near each curve is the corresponding value of $T/(\gamma_s P_0^2)$. The behavior of the curves near $M = 0$ is shown in the inset: $M = 0$ remains a local minimum of the free energy and thus a metastable state.

However, at $M = 0$ there is always a local minimum of the free energy: the cw is always a metastable state (see Fig. 1). To study this local minimum, we expand the free energy given in Refs. 13 and 14 for small M and small T, obtaining

$$F(M) - F(0) \approx NTM^2 - \frac{N\gamma_s P_0^2}{3} M^4. \quad (8)$$

We neglected all the powers of M higher than 4 as well as the TM^4 term. γ_s is the self-amplitude-modulation coefficient. The height of the barrier confining the system in a cw state, the maximum of relation (8), is $3NT^2/(4\gamma_s P_0^2)$, and to cross this barrier, the temperature T should be higher than that barrier. We therefore obtain the condition $\gamma_s P_0^2 > 3W/8$, which is in close accordance with relation (7).

Relation (7) provides only the order of magnitude of the self-starting power. The precise condition depends on the precise form of the equations of motion. Moreover, since pulsation starts from an intensity fluctuation, the precise self-starting condition depends on how long one is willing to wait for a sufficiently large intensity fluctuation, and the formula itself must be probabilistic.

A remarkable conclusion of relation (7) is that the power of the noise needed to inhibit modulation instability is very small. The noise-to-signal ratio W/P_0 is proportional to the nonlinear coefficient of the system times the average optical power (γP_0), which reflects the fact that noise does not compete with the intracavity energy. It competes with the interaction induced by the nonlinearity, which is very small.

It is interesting to compare relation (7) with that obtained from the decoherence time model.[7] Comparing the last term of Eq. (6) with Eq. (3′) from Ref. 7 suggests identifying $2P_0/W$ as the decoherence time. In this case the self-starting condition of Ref. 7 coincides with relation (7) up to a factor of ln N, which is subleading in N and probably falls beyond the approximations we made in this Letter. This also provides a formula for the decoherence time caused by additive noise and in particular by quantum noise.

Quantum noise is well modeled by white Gaussian Langevin terms and is probably the most fundamental source of noise in lasers. It is interesting to evaluate whether quantum noise alone can account for the mode-locking threshold. The total power of the noise of an amplifier is $\hbar\omega B(G - 1)$,[17] where ω is the optical frequency, B is the bandwidth, and G is the total gain per pass. This power is usually multiplied by an enhancement factor when the amplifier is saturated. Taking typical values for an erbium amplifier, with $B = 10^{12}$ Hz, we obtain $W \approx 7$ nW. With γ typically being 2 $\mathrm{W^{-1}\,km^{-1}}$ for fibers and the intracavity length being 10 m, relation (7) predicts a threshold of $P_0 \approx 1$ mW, which is a plausible result in fiber lasers. This example demonstrates how a small noise-to-signal ratio is sufficient for stabilizing a cw operation.

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15. For simplicity of notation Γ_m is defined here differently from Ref. 13; here the Γ_m have a power of unity.
16. This is obtained from Stratonovich calculus. Intuitively, $\mathrm{Re}[a_m^* \sqrt{(\epsilon W)/(2N)}\,\Gamma_m]$ is the rate at which the noise supplies energy to the mth mode, which is $(\epsilon W)/(2N)$, the power of the noise. The summation on m adds a factor of N.
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