Phase Transition Theory of Many-Mode Ordering and Pulse Formation in Lasers

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A novel theory for the ordering of many interacting modes in lasers is presented. By exactly solving a Fokker-Planck equation for the distribution of waveforms in the laser in steady state, equivalence of the system to a canonical ensemble is established, where the role of temperature is taken by amplifier noise. Passive mode locking is obtained as a phase transition of the first kind and threshold is calculated, employing mean field theory backed up by a numerical study. For zero noise, compliance with the existing noiseless theory is shown.

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When a laser operates in a multimode regime, nonlinearities in its medium can induce interactions between the cavity modes. When the number of modes is large (10^2–10^9 in long lasers), the laser becomes a system of many interacting degrees of freedom. Among the various nonlinearities which can be found in lasers, there is one, saturable absorption, which gives rise to spectacular behavior of the laser: formation of long-ranged order in the mode population by aligning their temporal phases together. In the time domain this manifests itself as pulsation. This behavior of the modes is called ''mode locking'' [1], and when it is achieved by a saturable absorber it is called ''passive mode locking'' (PML). PML has been receiving increasing attention over the last decade, due to its scientific and technological importance, mainly as a tool for producing the shortest light pulses ever obtained by man. It is a well-studied branch of laser physics.

Although saturable absorbers favor pulses, pulsation is achieved only when the optical power reaches a certain threshold (beside what is needed for basic oscillations), and the absorbers' nonlinearity is ''felt'' strongly enough. The emergence of pulses upon reaching the threshold is known to be abrupt [2]. This interesting threshold behavior of PML received far less attention than PML itself and has been addressed rather sporadically. The existence of a threshold led researchers to seek for a mechanism in the laser which opposes mode locking and needs to be overcome. Several candidates have been suggested for this mechanism [2], and yet it seems that the threshold behavior of PML, and, in particular, its abruptness, are not well understood.

In this Letter we present a novel statistical-mechanical theory for the ordering of modes. Lasers are systems far from thermal equilibrium, but they often reach a steady state. A common tool for finding their steady-state statistical distribution is the Fokker-Planck equation, and it has provided many theoretical predictions for the statistics of laser light which have been experimentally verified [3,4]. In this Letter we present an exact solution for the steady-state distribution of the modes in a PML laser, which turns out to be the Gibbs distribution, with generalized temperature and energy. Thus an equilibriumlike statistical distribution emerges in a nonequilibrium system.

The occurrence of the Gibbs distribution paves the way for discussing PML as an order-disorder phase transition: effort is needed for mode locking to occur because being ordered, mode-locked configurations are rare. The quantitative measure of their rareness is entropy, and it is the balance between entropy and (a generalization of) energy which governs PML. An important result of this study is that the formation of pulses is abrupt since it is a phase transition of the first kind. This is a striking new aspect of PML of lasers. We derive a simple formula for the PML threshold condition which is the first, to our knowledge, to express that threshold in terms of known (rather than specially tailored) parameters. The idea of relating phase-transition theory and laser physics has been raised before [3–8], regarding mainly the lasing threshold. We are the first however, to our knowledge, to identify PML as a phase transition.

The equation of motion we begin with is the Landau-Ginzburg–like equation often referred to as ''the master equation'' [1]. Noise due to spontaneous emission is represented by white Gaussian additive Langevin forces [9]. The Gibbs distribution is studied numerically via Monte Carlo simulation and analytically through mean field theory (MFT).

The main ingredient in the evolution of an optical narrow-band wave packet is translational motion with the group velocity v_g. Because of dispersion and nonlinearities the propagating wave packet undergoes some distortion, which is slow compared to the optical frequency, provided that the optical band is relatively narrow and nonlinearities are small. It is this slow waveform-shaping dynamics that we are interested in.

The latter dynamics is divided into two types: Hamiltonian (conservative, refractive) and dissipative. The former includes dispersion and the effect of an intensity dependent refractive index (the Kerr effect). The latter includes saturable absorption and gain, which are necessary ingredients for modeling a PML laser.

In order to make the discussion specific we restrict it to a ring laser (of length L), where light is allowed to propagate

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in one direction only. A customary way to examine the evolution of the optical waveform is by taking "snapshots" of it every round-trip time L/v_g. The "center" of the wave packet is thus located at the same position at every snapshot, and the evolution of the waveform can be traced. If the waveform changes only slightly between the snapshots, the evolution can be approximately given by a differential equation, which is commonly referred to as the master equation [1]. In terms of the slowly varying mode amplitudes defined through

$$E(z,t) = \sum_m [a_m(t) e^{-i(\omega_0 + \Omega m)t}] e^{i(k_0 + \Delta k m)z} + \text{c.c.}, \tag{1}$$

it reads

$$\dot{a}_m = (i\gamma_d - \gamma_g) m^2 a_m + (i\gamma_k + \gamma_s) \sum_{j-k+l-m=0} a_j a_k^* a_l + g a_m, \tag{2}$$

where $\Delta k = 2\pi/L$, $\Omega = v_g \Delta k$, k_0 and ω_0 are, respectively, the central wave number and frequency of the optical wave packet, the dot stands for a time derivative, γ_d and γ_g characterize dispersion and net gain profile (both are usually parabolically approximated), γ_k and γ_s represent the nonlinearities in refractive index and absorption, and finally g is the net gain at ω_0. Equation (2) is actually a slightly modified version of the one explained in Ref. [1]: we have changed the notation, performed a Fourier transform, and we consider the waveform as a function of space rather than time.

If $\gamma_g = 0$, $g = 0$, and $\gamma_s = 0$, Eq. (2) describes Hamiltonian dynamics. The Hamiltonian is

$$\mathcal{H}_R = -\gamma_d \mathcal{R} - \gamma_k \mathcal{Q},$$

with

$$\mathcal{Q} = \frac{1}{2} \sum_{j-k+l-m=0} a_j a_k^* a_l a_m^*; \qquad \mathcal{R} = \sum_m m^2 a_m a_m^*,$$

with a_k for the coordinates and ia_k^* for their adjoint momenta. a_k and a_k^* are treated as independent variables upon differentiation. Like $\mathcal{H}_R$ itself, the optical power [10] $\mathcal{P} = \sum_m a_m a_m^*$ is a constant of motion under Eq. (2).

If γ_g, g, and γ_s are nonzero, the dynamics is no longer Hamiltonian and $\mathcal{P}$ is no longer preserved. In lasers, the gain is signal dependent, decreasing as the optical signal intensifies. This mechanism is the one which stabilizes optical power in lasers, and, in particular, in passively mode-locked lasers [11]. In order to obtain PML it is crucial to pick an amplifier with a slow gain response. Otherwise, gain saturation tends to cancel absorption saturation. For a narrow-band wave packet γ_g can be approximated as constant [11], and g is usually modeled by $g = g_0/(1 + \mathcal{P}/P_{\text{sat}})$ (P_{sat} is the saturation power of the amplifier). Although this and even a more faithful model of an optical amplifier can be easily incorporated into our formalism [12], we present here a simpler model for gain saturation, which we claim preserves its essence: At any instant, g assumes the value needed for maintaining $\mathcal{P}$ *strictly constant* ($\mathcal{P} = P_0$). This simplifies the theory, and although in practice $\mathcal{P}$ can vary within some range, clearly uniform amplification does not affect the order-disorder (phase) properties of the modes and hence is immaterial to understanding PML. To calculate the required g we multiply Eq. (2) by $2a_m^*$, sum over m, and take the real part of the equation. On the left-hand side would be $\dot{\mathcal{P}}$, which is required to be zero. This determines g:

$$g = \frac{\gamma_g \mathcal{R} - 2\gamma_s \mathcal{Q}}{\mathcal{P}}, \tag{3}$$

which substituted back to Eq. (2) yields an equation which can be written as

$$\dot{a}_m = \frac{\partial(\mathcal{H}_R - i\mathcal{H}_I)}{i\partial a_m^*}; \tag{4}$$

$$\mathcal{H}_I = -\gamma_s P_0^2 \frac{\mathcal{Q}}{\mathcal{P}^2} + \gamma_g P_0 \frac{\mathcal{R}}{\mathcal{P}}.$$

Equations (4) are the Hamilton equations of motion, only the Hamiltonian now has an imaginary part $-i\mathcal{H}_I$, which is responsible for the "generalized friction" acting on the system. Equations (4) will therefore still preserve $\mathcal{P}$ but will not in general preserve the Hamiltonian anymore.

Like in some of the PML theories, we consider noise to be the mechanism opposing mode locking. The primary source of noise in lasers is spontaneous emission, which is commonly [9] modeled by a white Gaussian "Langevin force." We henceforth restrict our discussion to a system with a purely imaginary Hamiltonian. We discuss this limitation at the end of the article. Changing variables to a_m^R and a_m^I, the real and imaginary part of a_m, and adding Langevin forces renders Eqs. (4):

$$\dot{a}_m^R = -\frac{\partial \mathcal{H}_I}{\partial a_m^R} + \Gamma_m^R; \qquad \dot{a}_m^I = -\frac{\partial \mathcal{H}_I}{\partial a_m^I} + \Gamma_m^I \tag{5}$$

with Γ_m^R and Γ_m^I being the real and imaginary parts of the complex Langevin force. We assume the Γs to be Gaussian, white, statistically independent and of the same spectral power $2T$. We refer to T as temperature, since it will be shown to play this role. This parameter can be related to the fundamental properties of the optical amplifier in use. It should be commented that introducing such noise violates the conservation of $\mathcal{P}$. Regaining conservation of $\mathcal{P}$ can be achieved by projecting out of the Γs the single vector component responsible for the change in $\mathcal{P}$. This will introduce correlations of order $1/N$ (N is the number of the modes in the band) between the Γs, which we shall neglect.

From Eqs. (5) we can see that these nonlinear Langevin equations satisfy the "potential condition": The time derivative of each coordinate equals to the derivative of a "potential" with respect to it, plus white Gaussian noise. Therefore [4], the steady-state solution for the corresponding Fokker-Planck equation, in other words, the steady-state distribution of the as, is the Gibbs distribution:

$$\rho(a_1, \ldots a_N) = \frac{e^{-\mathcal{H}_I/T}}{Z}; \qquad Z = \int e^{-\mathcal{H}_I/T} da_1 \cdots da_N, \tag{6}$$

where Z is a normalization coefficient (partition function). Since $\mathcal{H}_I$ is unaffected by scaling the as, in the calculation of Z and all statistical averages it is enough to integrate over the as only on the sphere

$$\mathcal{P} = P_0. \tag{7}$$

The distribution of the waveforms in a PML laser has been subject to experimental and theoretical studies, mainly in context of the noise properties of the PML laser output. Analytically, however, only the lowest moments of this distribution are known [9]. The latter are obtained through linearization of the equation of motion. Equation (6) is an *exact* solution for the distribution, obtained from the full nonlinear equation of motion.

For the noiseless case ($T = 0$), the minimum of $\mathcal{H}_I$ is reached. By simple differentiation and using Eq. (3) we find that at the minimum of $\mathcal{H}_I$ the waveform satisfies

$$\gamma_g \frac{\partial^2 \psi}{\partial \zeta^2} + \gamma_s |\psi|^2 \psi + g\psi = 0; \qquad \psi(\zeta) = \sum_m a_m e^{im\zeta}.$$

This is a well-known equation [1], which has the secant hyperbolic solution confirmed experimentally [13].

We now proceed to $T > 0$. The interaction in $\mathcal{H}_I$ is long ranged. Each "particle" interacts with all others. This means that MFT becomes exact in the thermodynamic limit [14]. We now present two variants of MFT, restricting ourselves in this Letter to the case of $\gamma_g = 0$. Of all distributions $\rho(a_1, \ldots, a_N)$, $\rho \propto e^{-\mathcal{H}/T}$ is the one which minimizes the Helmholtz free energy functional:

$$F_\rho = \langle \mathcal{H} \rangle_\rho - T\langle S \rangle_\rho = \langle \mathcal{H} \rangle_\rho + T\langle \ln(\rho) \rangle_\rho, \tag{8}$$

where $\langle \cdot \rangle_\rho$ denotes average with respect to ρ. MFT is merely limiting our search for a minimizer to a certain class of distributions: those where all the as are identically independently distributed: $\rho(a_1, a_2, \ldots, a_N) = \rho(a_1)\rho(a_2) \cdots \rho(a_N)$. We favor this approach to MFT over the "effective field" approach since $\mathcal{H}_I$ has no translation invariance: Even as N goes to infinity, still as in the middle of the band appear in roughly by 1.5 more quartets than as on the edge of the band do. This makes the usual effective field approach difficult, since the effective field felt by a particle depends on its location.

We further limit our search for a minimizer to a yet narrower class of distributions: $\rho(a) = \rho_r(r)\rho_\theta(\theta)$, with r and θ for the modulus and argument of a. In $\mathcal{H}_I$ there are $\frac{2}{3}N^3 + O(N^2)$ terms with all the indices j, k, l, m different, and $O(N^2)$ terms with two or more indices equal. Neglecting the latter ones, we obtain

$$\langle \mathcal{H}_I \rangle \approx -\frac{\gamma_s}{3} N^3 \mu^4 |M|^4, \tag{9}$$

where

$$M = \int_0^{2\pi} \rho_\theta(\theta) e^{i\theta} d\theta; \qquad \mu = \int_0^\infty r\rho_r(r) r dr. \tag{10}$$

Since $\langle \mathcal{H}_I \rangle$ depends on $|M|$ only, M can be assumed real without limiting generality. Now, the entropy is given by

$$\langle S \rangle = -N\int_0^{2\pi} \rho_\theta \ln(\rho_\theta) d\theta - N\int_0^\infty \rho_r \ln(\rho_r) r dr. \tag{11}$$

If we insist on strictly satisfying the constraint of Eq. (7), we have no choice but to set

$$\rho_r(r) = \sqrt{N/P_0}\ \delta(r - \sqrt{P_0/N}),$$

so $\mu = \sqrt{P_0/N}$. Then the second term in Eq. (11) is dropped. Since $\langle \mathcal{H}_I \rangle$ depends only on M, before minimizing Eq. (8) we can maximize the entropy alone with respect to ρ_θ with M fixed. By the principle of maximum entropy [15], its maximum is achieved at

$$\rho_b(\theta) = \frac{e^{b\cos(\theta)}}{2\pi I_0(b)}, \quad \text{where } M = \frac{I_1(b)}{I_0(b)}.$$

The Is are the modified Bessel function of the first kind, and b is a real number determined by M. The resulting F, which minimized with respect to b leads to a first order phase transition (see Fig. 1), is

$$\frac{F(b)}{N} = -\frac{P_0^2 \gamma_s}{3} \frac{I_1^4(b)}{I_0^4(b)} + T\left(b\frac{I_1(b)}{I_0(b)} - \ln[2\pi I_0(b)]\right). \tag{12}$$

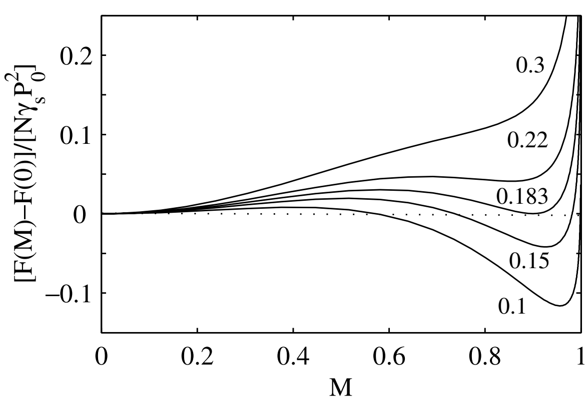

FIG. 1. Plot of the free energy per degree of freedom as function of M for several temperatures, for the version of MFT where $|a|$ is constant [Eq. (12)]. The number near each curve is the dimensionless magnitude $T/(\gamma_s P_0^2)$. This theory predicts the transition to occur at $T/(\gamma_s P_0^2) \approx 0.183$. Note the metastable states below and above the phase-transition temperature, typical to phase transitions of the first kind, which are associated with supercooling and superheating.

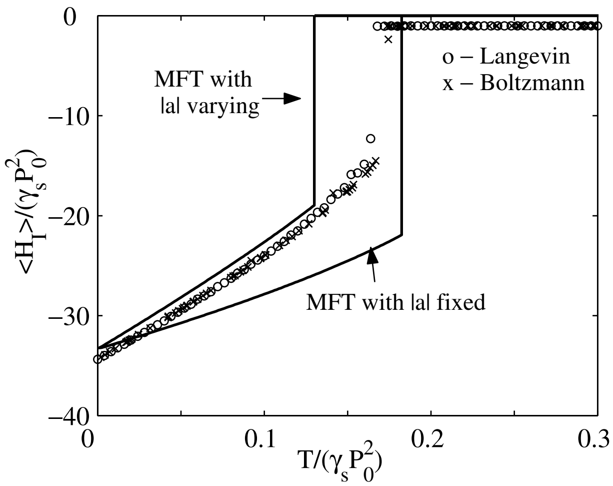

FIG. 2. Plot of $\langle \mathcal{H}_I \rangle$ as a function of T, both divided by $\gamma_s P_0^2$ to form dimensionless parameters. For $N = 100$ numerical simulations of the Langevin Eqs. (5) are demonstrated to give the same results as a Monte Carlo simulation for the Gibbs distribution Eq. (6). Both versions of the MFT are shown as well. Notice that $\mathcal{H}_I/(\gamma_s P_0^2) = \mathcal{Q}/\mathcal{P}^2$ and thus it is roughly inversely proportional to the optical pulse width—a natural order parameter.

In Fig. 2 we have plotted $\langle \mathcal{H}_I \rangle$ as a function temperature, obtained through both MFT and numerical simulations. For the latter, we took $N = 100$ and performed both direct numerical simulation of the Langevin Eqs. (5) and a Monte Carlo simulation for the Gibbs distribution Eq. (6). While the numerical simulations fit well with each other, MFT misses them somewhat, predicting the phase transition to occur at $T/(\gamma_s P_0^2) \approx 0.18$ instead of the value ~ 0.17 predicted by the simulations. This is so because the MFT underestimates entropy, fixing half of the degrees of freedom. Nevertheless, the error in the prediction of the transition temperature is minor (see Fig. 2), which reflects the fact that the main contribution to the order-disorder transition comes from the phases rather than from the moduli. Naturally, however, it gives half the "heat capacity" ($\frac{\partial \langle \mathcal{H}_I \rangle}{\partial T}$) at low temperatures.

An approach that gives a better description in terms of MFT at low temperatures is by relaxing the power constraint of Eq. (7), requiring it to hold only on the average: $N \int_0^\infty r^2 \rho_r(r) r dr = P_0$. Here we do not bring this version of the MFT apart from its result, which is plotted at Fig. 2. Contrary to the former version of MFT, this one overestimates entropy, since it allows the system extra freedom. This lowers the transition temperature. However, relieving the power constraint to some extent only makes the model more realistic.

The relation $T_c \propto P_0^2$ (where T_c is the phase-transition temperature) which follows from the above discussion, was established for the case $\gamma_g = 0$ and for a finite number of modes. This defines a rectangular gain profile. We note that when $\gamma_g > 0$ and the band of modes is not truncated, that is, a parabolic gain profile, it is easy to show that $T_c \propto P_0^{3/2}$.

By presenting a theory for PML which on one hand fully relies on a well-established model and on the other hand shows exact obedience to the Gibbs distribution, we harness the entire power of phase-transition theory for understanding PML of lasers. A wide variety of effects, such as hysteresis—supercooling and superheating, as well as analogs of droplet formation and latent heat, can now be expected to be found and measured in pulse lasers.

In this Letter we did not analyze dispersion and the Kerr effect, which are not negligible in some of the configurations used for PML. It is easy to show that our theory holds as is with dispersion and the Kerr effect for a specific choice of their coefficient—where $\mathcal{H}_R$ is proportional to $\mathcal{H}_I$. This extension of our theory includes the important case of soliton lasers [1]. Beyond that, at this stage we can comment that numerical studies indicate that adding these effects does not change the qualitative behavior and, in particular, the phase transition still exists. A phase transition of the first kind is a dramatic singularity which is not so easily affected when the model is slightly altered.

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