

Light-Mode Condensation in Actively-Mode-Locked Lasers

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We show that the formation of pulses in actively mode-locked lasers exhibits in certain conditions a transition of the laser mode system to a light pulse state that is similar to Bose-Einstein condensation (BEC). The study is done in the framework of statistical light-mode dynamics with a mapping between the distribution of the laser eigenmodes to the equilibrium statistical physics of noninteracting bosons in an external potential. The light-mode BEC transition occurs for a mode-locking modulation that has a power law dependence on time with an exponent smaller than 2.

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Active and passive mode locking of lasers are the main methods for obtaining ultrashort light pulses, that nowadays can reach the few femtosecond regime [1,2]. They are based on locking the phases of many axial (longitudinal) modes of a laser that can in certain cases span over most of the visible frequency band. While the mode locking has been traditionally viewed as a sort of light kinematical mechanism, we have developed an approach [3] that treats the many light-mode system with noise as a statistical mechanics theory. An important ingredient of it arises from the understanding that noise that takes the role of temperature in thermodynamic systems must be treated nonperturbatively. The outcome is statistical light-mode dynamics (SLD), a powerful theory for the discovery and study of points of sharp changes in the global structure of the optical waveform, that have the significance of thermodynamic phase transitions. It was found, for example, theoretically and experimentally [3–6], that passive mode locking is nothing but a first-order phase transition of the modes from random phase orientations to an ordered phase (pulses). We also applied SLD to active mode locking (AML) [7]. In an exact solution, valid in the parameter regime of flat gain filtering profile, no dispersion and harmonic modulation, the system was mapped to the Berlin-Kac spherical model [8] that lacks phase transitions in one dimension. Therefore, in that regime, there is no global ordering of the modes for the same reasons that prevent ordering in one-dimensional short-range interacting systems, and as a consequence, the pulses in that case are far less than optimal.

In the present Letter, we study AML that includes parabolic gain filtering and dispersion, where the gain modulation has a power law dependence on time with an exponent smaller than 2. We then find that the mode-locking phase transition occurs, and takes the form of power condensation, wherein the basic AML laser pulse eigenmode carries a macroscopic fraction of the total cavity power, in a full analogy to the macroscopic occu-

pation of the ground state in Bose-Einstein condensates (BEC) in an external potential [9]. BEC has been experimentally demonstrated in recent years in systems of cold atoms [10]. Still, it has been often observed that the condensation is a wave phenomenon, not limited to the quantum mechanics of bosons. In relation to optics, condensation has been theoretically attributed to light in a weakly nonlinear medium [11], and in random media [12]. Here, it is shown to occur in a laser system that can be easily studied in experiments. Indeed, we are in the midst of an experimental study with preliminary results that show for the first time laser light BEC.

Active mode locking is obtained by a periodical modulation of the laser, for example, via the effective gain, with a frequency that equals the frequency difference between consecutive axial modes of the laser. If the modulation is sinusoidal, as commonly used, it produces two sidebands of each mode, that may eventually lock the mode phases and lead to the formation of pulses. However, other forms of gain modulation can also be used, and these will be investigated here. The basic mechanism that determines the pulse width in AML is the competition between the production of sidebands by the gain modulation that tends to make the pulse narrower and the gain filtering that stems from the finite amplifier bandwidth that limits the spectral width of the pulse [13,14].

The effect of noise can be incorporated in the “master equation” approach [14] by adding a Langevin noise source term. The effect of classical and quantum noise has been considered in several papers since the 1970s [15–18] as a small perturbation of the noiseless pulse solution of the master equation. However, as was shown [3], in spite of its small amplitude, the perturbation approach is in general not justified, since the accumulation of noise in the whole cavity creates a continuum background that carries a significant fraction of the total cavity power and competes with the pulse. The resolution of this issue is achieved with SLD.

When applied to the nonlinear interaction of passive mode locking, the SLD analysis was solved in a mean-field theory that becomes exact in the thermodynamic limit, where the number of modes approaches infinity, and the onset of mode locking is a first-order phase transition [19,20]. The SLD analysis of AML is even simpler, since, the equation being linear, the SLD steady state can be expressed explicitly using the eigenmodes of the dynamical linear operator. It turns out that in the usual harmonic modulation case, even a small amount of noise may destabilize AML pulses, prohibiting the possibility of an AML transition altogether. This finding does not imply that AML pulses cannot be formed, but it limits the coherence properties and implies that pulses are much broader than the lower bound provided by the noiseless theory [13,14]. In particular, the assumption that only the fundamental AML eigenmode has a significant amplitude is in general not justified.

The situation is very different if the AML modulation is a nonsmooth function of time with a power law singularity. Here, we show that the AML eigenmode (not axial mode) system undergoes a BEC phase transition when the overall laser power is increased, or noise ("temperature") is decreased. Unlike passive mode locking, the transition is continuous, characterized by the buildup of a macroscopic power fraction in the fundamental eigenmode. This is precisely the mechanism that describes the onset of BEC in noninteracting boson systems when the temperature is decreased or the density is increased beyond the condensation point. The transition is observable as a qualitative change in the laser waveform as a function of the system parameters that becomes sharper when the number of AML eigenmodes increases, as shown in Fig. 1.

The standard AML master equation with an additive noise source is [14]

$$\frac{\partial \psi(t, \tau)}{\partial \tau} = \hat{O}(t)\psi(t, \tau) + g\psi(t, \tau) + \Gamma(t, \tau) \quad (1)$$

where $\psi(t, \tau)$ is the electric field envelope, and t and τ are the short and long time variables, describing the dynamics of ψ within a single round-trip time τ_R , and over the course

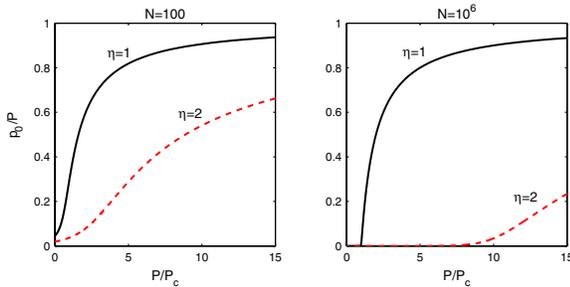


FIG. 1 (color online). The fraction of power in the first eigenmode p_0/P versus the normalized total power P/P_c for $\eta = 1$ (solid black line) and $\eta = 2$ (dashed red line). The left figure is calculated for $N = 100$, and the right for $N = 10^6$, shows for $\eta = 1$ a sharp BEC transition at $P = P_c$, and for $\eta = 2$ a gradual population buildup.

of many round-trip times, respectively [14,21]. The operator $\hat{O}(t)$ is

$$\hat{O}(t) = (\gamma_g - i\gamma_d)\frac{\partial^2}{\partial t^2} - V(t). \quad (2)$$

γ_g and γ_d are the gain filtering and dispersion coefficients, respectively, and g is the saturated net gain. $V(t)$ is the time-dependent loss-gain modulation. It is a periodic function with period τ_R , the cavity round-trip time, and we assume that its scale of variation is τ_R . Denoting the point at which V achieves its minimal value by 0, we will consider gain signals with a power law dependence $V(t) \sim M|\frac{t}{\tau_R}|^\eta$, for small t . The generally used sinusoidal modulation corresponds to $\eta = 2$, that otherwise characterizes any generic smooth minimum. Values of η that are not even positive integers, including the case of main interest here $0 < \eta < 2$, characterize a nonsmooth behavior of $V(t)$ at its minimum. Γ , an additive noise that originates from spontaneous emission and other possible internal and external sources, is modeled by a centered white Gaussian process with covariance $2T$ per unit length, $\langle \Gamma(t, \tau)\Gamma(t', \tau') \rangle = 2TL\delta(t-t')\delta(\tau-\tau')$, and $\langle \Gamma(t, \tau) \rangle = 0$.

Since the waveform ψ is the solution of a Gaussian forced linear equation, its steady state distribution is also Gaussian, and can be explicitly expressed in terms of the eigenmodes of \hat{O} , $u_n(t)$, $n = 0, 1, 2, \dots$, with corresponding eigenvalues $-\varepsilon_n$; we assume that the eigenmodes are a complete set, ordered so that $\text{Re}\varepsilon_n$ is increasing. The eigenfunctions, normalized so that $\int \frac{dt}{\tau_R} |u_n(t)|^2 = 1$, are orthogonal only for the zero dispersion case $\gamma_d = 0$.

In terms of the eigenmode expansions $\psi(t, \tau) = \sum_n \psi_n(\tau)u_n(t)$, $\Gamma(t, \tau) = \sum_n \Gamma_n(\tau)u_n(t)$, the master equation (1) becomes

$$\frac{d\psi_n(\tau)}{d\tau} = (g - \varepsilon_n)\psi_n(\tau) + \Gamma_n(\tau), \quad (3)$$

and it follows that $\psi_n(\tau)$ are colored Gaussian processes whose steady state variance $\langle |\psi_n(\tau)|^2 \rangle = \frac{T}{\text{Re}\varepsilon_n - g}$ is equal to the mean power p_n in eigenmode n .

The steady state distribution is therefore determined by the spectrum ε_n of the low-lying eigenmodes, whose qualitative properties are as follows: If the temporal extent of the n th eigenmode is t_n , then $|\varepsilon_n| \sim M(t_n/\tau_R)^\eta$; on the other hand, the eigenmode frequency is t_n/n so that we can make also the estimate $|\varepsilon_n| \sim |\gamma_g - i\gamma_d|(n/t_n)^2$ so that

$$(t_n/\tau_R) \sim (n^2/N^2)^{1/(2+\eta)}, \quad (4)$$

where $N = (M\tau_R^2/\gamma_g)^{1/2}$. The dimensionless parameter N may be interpreted as the number of eigenmodes affected by the modulation. It is also a measure of the degree of pulse compression achieved by the mode-locking action, so we will assume that $N \gg 1$ in the thermodynamic limit.

Our main interest is in calculating the power in the fundamental eigenmode p_0 and comparing it to the total power P . However, the analysis is hampered by the fact

that the $\psi_n(\tau)$ processes are not independent because of the chromatic dispersion. For this reason, we assume provisionally that $\gamma_d = 0$ and return to the general case later, to argue that the basic results remain unchanged in the presence of dispersion. When dispersion is neglected, the different eigenmode amplitudes are uncorrelated, the eigenvalues are real, and then

$$P = \left\langle \int |\psi(t)|^2 \frac{dt}{\tau_R} \right\rangle = \sum_n \frac{T}{\varepsilon_n - g}. \quad (5)$$

The saturated net gain g depends on P , and Eq. (5) should be solved for P to give the distribution of power within the normal eigenmodes. Nevertheless, it is useful to think of the ratio P/T as given, and that it determines g by (5), and through it the entire distribution.

It follows from Eq. (5) that when P/T increases, the net gain must also increase. If $|g| \sim \varepsilon_{N_0}$, then the power in the first N_0 modes is roughly constant, and using Eq. (4), we can estimate the total power in the first N_0 modes by

$$P_{N_0} = (T/M)N^{(2-\eta/2+\eta)}. \quad (6)$$

When $\eta > 2$, P_{N_0} is a decreasing function of N_0 , and a broad power distribution is possible for any P , but for $\eta < 2$, increasing P/T lowers P_{N_0} and for P/T large enough, all additional power must go into the fundamental mode.

The analysis is based on the Bohr-Sommerfeld quantization rules for the ε_n 's [22], whose validity includes our domain of interest $N \gg 1$, and imply that

$$\sum_n \frac{1}{\varepsilon_n - g} \rightarrow \int \frac{\rho(\varepsilon)d\varepsilon}{\varepsilon - g}, \quad (7)$$

where

$$\rho(\varepsilon) = (1/2\pi) \int dt d\Omega \delta(\varepsilon - (\gamma_g \Omega^2 + V(t))) \quad (8)$$

is the density of eigenstates of the operator \hat{O} . For $\varepsilon \ll M$, we can use the power law asymptotics of $V(t)$ to write

$$\rho(\varepsilon) = c_\eta \frac{N}{M} \left(\frac{\varepsilon}{M}\right)^\xi, \quad \xi = \frac{1}{\eta} - \frac{1}{2} \quad (9)$$

with $c_\eta = \frac{1}{4\pi} \int_0^1 \frac{ds}{\sqrt{1-s^\eta}}$.

The solution of Eq. (5) is therefore entirely analogous to the calculation of the chemical potential of a system of noninteracting bosons in a trap with potential V [9]. Although the integration kernels are different in the two problems, the question of condensation depends only on the low energy asymptotics of the kernel, leading to the same result that condensation takes place above a threshold power for $\eta < 2$, where the integral in (7) converges in the infrared. In the case of trapped bosons, this leads to BEC, whereas in the laser system, it leads to transition to a single AML eigenmode.

Thus, Eq. (5) can be written as

$$P = \frac{T}{\varepsilon_0 - g} + T \int_0^M \frac{\rho(\varepsilon)d\varepsilon}{\varepsilon - g}, \quad (10)$$

and condensation occurs for $\eta < 2$ whenever $P > P_c = T \int \frac{\rho(\varepsilon)d\varepsilon}{\varepsilon}$. In the condensate phase, $p_0 \sim P - P_c$ compared with $p_n \sim P_c N^{-\xi}$ for $n > 0$ so that the power distribution obtains a giant $n = 0$ component, in analogy with the macroscopic ground-state occupation of BEC. As a consequence, the waveform develops a strong u_0 component; this is, the narrowest possible AML pulse that saturates the Kuizenga-Siegmann limit [13]. In contrast, without mode condensation, the pulse width is determined as an average of many low-lying modes, and the pulse width shrinks slowly as power is increased.

The AML transition to the condensate state is sharp, and $p_0 = P - P_c$ only in the thermodynamic limit $N \rightarrow \infty$. For finite N , there is a gradual buildup of power in the ground-state eigenmode and $p_0 \sim P_c N^{-[\xi/(1+\xi)]}$ already for $P = P_c$. The width of the transition to condensate AML can be defined as the increment in P/T required to double p_0 , that is also of $O(N^{-[\xi/(1+\xi)]})$. The preceding ideas are illustrated for $\eta = 1$ in Fig. 1 where p_0/P is plotted versus P/P_c for two values of N . The power distribution was calculated for a sawtooth modulation function $V = M|t/\tau_R|$, $t \leq \tau_R/2$ (neglecting the small contribution of modes with $\varepsilon > M/2$) as a typical gain modulation profile that allows condensation, and two values of N . The graph for smaller value $N = 100$ (left panel) exhibits the smooth transition at P_c clearly, compared to the sharp transition that is apparent in the right panel calculated for $N = 10^6$.

The mode condensation is manifested by the creation of a strong narrow component in the waveform. A further direct consequence of the condensation is a sharp rise in the coherence time of the AML pulse. Below the condensation threshold, the pulse consists of an incoherent combination of a large number of modes so that its coherence time is shorter by N than the condensate pulse coherence time.

We turn to consider the borderline case $\eta = 2$. This case is of high interest as it corresponds to the standard practice of sinusoidal modulation. It also exhibits interesting AML properties since the infrared divergence of the integral in Eq. (7) is only logarithmic

$$\int \frac{\rho(\varepsilon)d\varepsilon}{\varepsilon - g} \sim -\frac{c_2 N}{M} \log \frac{|g|}{M}. \quad (11)$$

As a consequence, there is an exponential enhancement of the ground-state power

$$p_0/p_n \sim e^{(2\pi/c_2)(PM/NT)} \quad (12)$$

(for $n \geq 1$). For $\eta < 2$, the condensation threshold P_c is proportional to NT/M ; for cavity power values of this order of magnitude, the exponential factor in (12) is bounded so that no condensation occurs when $\eta = 2$, as claimed above. Nevertheless, a logarithmic enhancement in the pumping to $P \sim N \log N$ will make p_0 of the same order of magnitude as in the $\eta < 2$ condensate phase, achieving limiting width AML pulses. In contrast with

the true condensation, the crossover to ground-state dominance does not occur for a constant power per mode, and is therefore never sharp.

The AML properties for $\eta = 2$ are shown in Fig. 1, and compared with those exhibiting true condensation. For moderate value of N , there is little qualitative difference between the two cases, while for large N the condensate AML system shows a sharp buildup of power in the ground-state eigenmode at $P = P_c$, while the $\eta = 2$ system shows only a smooth increase of p_0 . The quasicondensation occurs for a much larger P , but since the transition power factor is logarithmic, it is still attainable for such a large value of N .

When $\eta > 2$, $\xi < 0$ and the integral in (7) diverges for small g as $|g|^\xi$. It follows that Eq. (10) can always be solved with a negative value of g , and mode condensation does not occur. In particular, when $\eta > 2$, the power needed to obtain the limiting width AML pulses is larger by a factor of $O(N^{-\xi})$ than that needed for $\eta < 2$.

We generalize our result for the case of nonzero chromatic dispersion γ_d , by showing that the power distribution depends on it very weakly, as long as it is not much larger than the gain filtering coefficient γ_g . This result relies on the validity of Eq. (10) in this case; to establish it, we express the solution of the master equation (1) as

$$\psi(t, \tau) = \int_{-\infty}^{\tau} e^{(\hat{O}-g)(\tau-\tau')} \Gamma(\tau') d\tau' \quad (13)$$

where $e^{(\hat{O}-g)\tau}$ is the operator exponential, whose position representation is the Green's function of the operator. The overall power can then be written as

$$\begin{aligned} P &= \left\langle \int |\psi(t)|^2 (dt/\tau_R) \right\rangle \\ &= \int (d\tau/\tau_R) d\tau' d\tau'' \langle e^{(\hat{O}^\dagger-g)(\tau-\tau')} \Gamma(\tau')^* e^{(\hat{O}-g)(\tau-\tau'')} \Gamma(\tau'') \rangle \\ &= 2T \int_{-\infty}^{\tau} d\tau' \text{Tr}(e^{(\hat{O}^\dagger-g)(\tau-\tau')} e^{(\hat{O}-g)(\tau-\tau')}) \end{aligned} \quad (14)$$

where Tr stands for operator trace. When $\gamma_d = 0$, O and O^\dagger commute so that the exponentials can be combined, and the integral over time carried explicitly in the last equality, giving the inverse of $2(\hat{O} - g)$, whose trace is expressed in terms of the eigenvalues in (5). This is not possible when $\gamma_d \neq 0$, but we can derive Eq. (10) directly under the assumption $N \gg 1$ using the Weyl operator calculus [23]. In the Weyl calculus, the trace of a product of operators is expressed as a phase-space integral of the product of their Weyl representations. The Weyl representation of \hat{O} is $O(t, \Omega) = -(\gamma_g - i\gamma_d)\Omega^2 + V(t)$, where Ω is the conjugate to t variable, and that of $e^{\hat{O}\tau}$ is approximately $e^{O(t, \Omega)\tau}$, unless τ is very large. Similarly, the Weyl representation of $e^{\hat{O}^\dagger\tau}$ is approximately $e^{O(t, \Omega)^*\tau}$. Therefore, for $N \gg 1$,

$$\text{Tr}(e^{(\hat{O}^\dagger-g)\tau} e^{(\hat{O}-g)\tau}) = (1/2\pi) \int dt d\Omega e^{2(\text{Re}O(t, \Omega)-g)\tau} \quad (15)$$

from which (10) follows immediately, as was to be shown.

Conclusion.—We have shown how BEC can be applied to laser mode locking, offering an immense improvement both in pulse shortening and in its coherence properties, at the same time exhibiting a new class of phase transitions in statistical light-mode dynamics.

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