Critical Behavior of Light in Mode-Locked Lasers

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Light is shown to exhibit critical and tricritical behavior in passively mode-locked lasers with externally injected pulses. It is a first and unique example of critical phenomena in a one-dimensional many-body light-mode system. The phase diagrams consist of regimes with continuous wave, driven parapulses, spontaneous pulses via mode condensation, and heterogeneous pulses, separated by phase transition lines that terminate with critical or tricritical points. Enhanced non-Gaussian fluctuations and collective dynamics are present at the critical and tricritical points, showing a mode system analog of the critical opalescence phenomenon. The critical exponents are calculated and shown to comply with the mean field theory, which is rigorous in the light system.

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Statistical light-mode dynamics (SLD) [1,2] that is based on statistical mechanics provides a powerful approach for the study of complex nonlinear light systems, while serving as a new statistical mechanics paradigm. It was developed to treat and solve long-standing questions in laser physics. A prime example is the laser pulsation threshold, a phenomenon in passive mode locking that attracted much attention [3]. Passive mode locking occurs when a saturable absorber element is placed in the cavity, driving the laser to generate extremely short light pulses. The transition mechanism from a continuous wave (cw) to pulsation of such lasers was shown via SLD [1,2] to be an intrinsic property of the many interacting mode system, ruled by the balance between the nonlinear interaction induced by the saturable absorber and the randomizing effect of noise. The theory, verified by experimental study, demonstrated [4] the existence of first order phase transitions between disordered (cw) and ordered (locked) mode phases, as the noise (“temperature”) or the laser power were varied. Thus noise alone stabilizes the cw state, showing a noise induced phase transition [5]. Concepts and ideas from phase transition theory and critical phenomena have been used in laser physics, and particularly in connection with modulation instability [6], but SLD is the first many-body *thermodynamic* theory of laser dynamics and mode locking.

In this Letter we report on findings of *critical* and *tricritical* phenomena in the many light-mode system. It is a first example of such behavior with light that also provides a physical realization of a strict one-dimensional many-body system. For criticality to appear in the laser mode system we add to it an external driving field which is an analog of the external magnetic field in magnets and the pressure in gas-liquid-solid systems [7]. It is achieved by injecting the laser with pulses from an external source, which in the simple case matches the repetition rate of the laser. When the injection is weak, the ordering phase transition persists, shifted to higher “temperature,” with a “parapulses” phase (pulses driven by the injection). However, the phase transition line terminates in a *critical point,* where the distinction between parapulses and spontaneous pulses disappears, similar to the vapor-liquid critical point. Thus, by increasing the injection, it is possible to obtain mode locking smoothly from cw. Near the critical point the system exhibits the familiar critical phenomena, including divergence of response coefficients, characterized by universal critical exponents, and non-Gaussian critical fluctuations, enhanced by a factor of $N^{1/4}$, where N is the number of active modes, compared to normal fluctuations. The latter could mean a 2 orders of magnitude fluctuation enhancement in practical systems. It is a light-mode system analog of the critical opalescence phenomenon.

When the injection repetition rate is higher than that of the laser, the high- and low-temperature phases are characterized by equal and unequal pulse powers, respectively, which cannot be smoothly connected. However, beyond a threshold injection level the transition becomes *continuous* rather than first order. The two phase transition lines meet at a *tricritical* point, often found in systems with interaction competing with external driving [8], around which tricritical behavior is observed, with its distinct set of universal exponents, and tricritical fluctuations enhanced by a factor of $N^{1/3}$.

Pulse injection has practical uses [9], and here we find additional aspects. For example, as in the vapor-liquid case, field induced “condensation,” that is in our case mode ordering and pulsation, can be sustained when the field is removed even below the threshold pumping.

We perform our theoretical study in the framework of the coarse-grained model of SLD, representing the cavity electric field envelope ψ by a single variable in an interval whose length is of the order of the pulse width. The derivation of the model from the passive mode-locking master equation [10] has been discussed before [1,2,4].

The dynamics of the field variable in interval number n is expressible as

$$\partial_t \psi_m = -\frac{\partial H}{\partial \psi_m^*} + g\psi_m + \eta_m(t). \quad (1)$$

The "Hamiltonian" H with injection strength h is

$$H = -N\,\mathrm{Re}\sum_{m=1}^{N}\left(\frac{\gamma}{2}|\psi_m|^4 + 2h_m\psi_m^*\right), \qquad (2)$$

where $N \gg 1$ is the number of active modes, γ is the coefficient of saturable absorption, and h_n is the external injection at site n. h_n is assumed to take nonzero values only at a small number n of intervals. g is the overall net gain, which can be assumed without loss of generality [2] to set the intracavity power $\|\psi\|^2 = \Sigma_n|\psi_n|^2$ to a fixed value P, in which case g becomes a Lagrange multiplier for the constraint. The random term η, representing noise from spontaneous emission and other sources, is modeled by a (complex) Gaussian white noise with covariance $\langle\eta_n^*(t)\eta_m(t')\rangle = 2T\delta_{nm}\delta(t-t')$.

The invariant measure of Eq. (2) is a Gibbs equilibrium distribution [1,11]

$$\rho[\psi] = \frac{1}{Z}e^{-H[\psi]/T}\delta(\|\psi\|^2 - P), \qquad (3)$$

with the partition function

$$Z = \int[d\psi][d\psi^*]e^{-H[\psi]/T}\delta(\|\psi\|^2 - P). \qquad (4)$$

As in previously studied cases of SLD [1,2,4], when $N \gg 1$ the invariant measure is concentrated on configurations where all but a finite number of the ψ variables are $O(N^{-1/2})$. Here the intervals where $\psi = O(1)$ are precisely the n intervals which are subject to external injection. The free energy $F = -T\log Z$ is then given by $F = N\min_{\psi_1,\ldots,\psi_n} f_n(\psi_1,\ldots,\psi_n)$, where

$$f_n = -\sum_{m=1}^{n}\left(\frac{\gamma}{2}|\psi_m|^4 + 2\,\mathrm{Re}h_m\psi_m^*\right) + T\log\left(P - \sum_{m=1}^{n}|\psi_m|^2\right). \qquad (5)$$

We establish Eq. (5) using the results of [2] for the partition function Z_0 for the $h = 0$ case,

$$Z_0 \sim \int d^2\psi e^{[N\gamma|\psi|^4 + NT\log(P-|\psi|^2)]/T} \qquad (6)$$

asymptotically for large N. We can proceed immediately to perform the integration in Eq. (4) over the $N - n$ intervals where $h = 0$

$$Z \sim \int d^2\psi\prod_m d^2\psi_m \times e^{-[H(\psi_1,\ldots,\psi_n) + N\gamma|\psi|^4 + NT\log(P - \Sigma_{m=1}^{n}|\psi_m|^2 - |\psi|^2)]/T}. \qquad (7)$$

The exponent in the integrand in Eq. (7) is proportional to the large parameter N, and therefore the integration is concentrated near the global minimum of the integrand. It is straightforward to verify that the minimum is always obtained when $\psi = 0$, implying that $F = N\min f_n$.

Moments of the pulse strength can be obtained in the standard manner by taking derivatives of the free energy. Alternatively we may obtain directly the probability distribution function of the pulse amplitudes by integrating out the $N - n$ unforced variables in Eq. (3) getting

$$\mathsf{P}(\psi_1,\ldots,\psi_n) \sim e^{Nf_n(\psi_1,\ldots,\psi_n)}. \qquad (8)$$

The width of the distribution tends to zero in the thermodynamic limit $N \to \infty$, which shows that the pulse amplitudes are thermodynamic observables.

We henceforth specialize to the case that all nonzero injection values are of the same magnitude h that is appropriate for the injection of pulses from a source with a repetition rate n times faster than that of the laser. For the minimization problem there is no loss of generality in assuming that the injection values are all real, since the minimizing ψ values have the same phase as the corresponding h values. By a simple rescaling of the variables the free energy f_n can be reduced to

$$f = -\sum_{m=1}^{n}\left(\frac{1}{2}x_m^4 + \mathfrak{h}x_m\right) - \mathcal{T}\log\left(1 - \sum_{m=1}^{n}x_m^2\right), \qquad (9)$$

where x_m's are real and positive. It follows from Eq. (9) that thermodynamics depends on only two dimensionless parameters, the reduced temperature $\mathcal{T} = \frac{T}{\gamma P^2}$ (the inverse of the interaction strength [1]) and reduced driving $\mathfrak{h} = \frac{2h}{\gamma P^{3/2}}$. The minimizers of f, $\bar{x}_m$, are related to the expectation values (in the invariant measure) of the pulse powers by $\bar{x}_m^2 = \langle|\psi_m|^2\rangle/P$.

The study of thermodynamics and critical behavior is now reduced to the analysis of the function f and its minima. We note that for any values of the parameters, minima can occur only at configurations where at least $n - 1$ of the $\bar{x}_m$'s are equal, with value $\bar{y}$, and the other minimizer, which we denote by $\bar{x}$ is greater than or equal to $\bar{y}$. Accordingly, the free energy is obtained from

$$f(x,y) = -\tfrac{1}{2}[x^4 - (n-1)y^4] - \mathfrak{h}[x + (n-1)y] - \mathcal{T}\log[1 - x^2 - (n-1)y^2]. \qquad (10)$$

It is instructive to consider the above results in **k** space for the modes a_k, the discrete Fourier transform of ψ_n. The Hamiltonian reads

$$H = -\frac{\gamma}{2}\sum_{j-k+l-m=0} a_j a_k^* a_l a_m^* - 2\,\mathrm{Re}\sum_k \tilde{h}_k a_k^*, \qquad (11)$$

where $\tilde{h}_k$ takes a nonzero value for k being integer multiples of n. Therefore, when $n > 1$, the set of modes consists of two types, with and without the presence of the external field. The Hamiltonian in Eq. (11) is analogous to the one of an antiferromagnet placed in an external homogeneous magnetic field [12], but with the roles of the interaction term and the driving term reversed. Namely, the driving acts on a subset of the modes, but the interaction tends to align all modes in the same amplitude and phase. The consequences of this competition are derived below. The free energy can be expressed in the mode representation using the mode amplitude expectation values $\langle a_f \rangle$ and $\langle a_u \rangle$ of those with (forced) and without (unforced) electric field, respectively. The relations are $x = \frac{1}{n}(\langle a_f \rangle + (n-1)\langle a_u \rangle)$ and $y = \frac{1}{n}(\langle a_f \rangle - 1\langle a_u \rangle)$, and the free energy is

$$f = -\frac{1}{2n^3}[\langle a_f \rangle^4 + (n-1)(n^2 - 3n + 3)\langle a_u \rangle^4 + 6(n-1)\langle a_f \rangle^2 \langle a_u \rangle^2 + 4(n-1)(n-2)\langle a_f \rangle \langle a_u \rangle^3] - \mathfrak{h}\langle a_f \rangle - \mathcal{T} \log\left(1 - \frac{1}{n}\langle a_f \rangle^2 - \frac{n-1}{n}\langle a_u \rangle^2\right). \quad (12)$$

For the analysis below we use the real space formulation. We consider first the case of $n = 1$, where there is a single pulse (the external field is applied on all modes), and f [Eq. (10)] depends on the single variable x. For zero and small values of $\mathfrak{h}$, f has a minimum x_1 near zero and, for small enough $\mathcal{T}$, another minimum $x_2 > x_1$ below 1. For such $\mathfrak{h}$ there is a threshold temperature $\mathcal{T}_1(\mathfrak{h})$ where $f(x_1) = f(x_2)$. As $\mathcal{T}$ is decreased through this line $\bar{x}$ jumps from x_1 to x_2 in a first order phase transition. When $\mathfrak{h} > 0$, the jump is between two pulse states, but the high-temperature phase pulses are driven pulses whose power decreases smoothly to zero when $\mathfrak{h} \to 0$. We term this phase "parapulse" because of its resemblance to the paramagnetic phase of magnets above the Curie point. For sufficiently strong $\mathfrak{h}$, on the other hand, f always has a single minimum $\bar{x}$ between zero and one, which decreases smoothly from one to zero as $\mathcal{T}$ is increased. Thus, the coexistence line $\mathcal{T}_1(\mathfrak{h})$ terminates at a critical point $(\mathfrak{h}_c, \mathcal{T}_c)$. The phase diagram is shown in the upper part of Fig. 1.

The lower part of Fig. 1 shows $\bar{x}$ as a function of $\mathcal{T}$ for several $\mathfrak{h}$ values. $\bar{x}$ undergoes a jump for $\mathfrak{h} < \mathfrak{h}_c$, and displays an infinite slope at the critical point—a manifestation of the critical divergence of the susceptibility. The critical point itself is characterized by the vanishing of the first three derivatives of f, which gives three polynomial equations for the three unknowns $\bar{x}_c$, $\mathfrak{h}_c$, and $\mathcal{T}_c$. The equations can be solved explicitly by radicals giving $\bar{x}_c \approx 0.53$, $\mathfrak{h}_c \approx 0.20$, and $\mathcal{T}_c \approx 0.34$.

One may define the usual critical exponents [7] β, γ, and δ by $(\bar{x} - \bar{x}_c)|_{\text{coexistence}} \sim (\mathcal{T}_c - \mathcal{T})^\beta$, $\chi = \partial\bar{x}/\partial\mathfrak{h} \sim |\mathcal{T} - \mathcal{T}_c|^{-\gamma}$, and $(\bar{x} - \bar{x}_c)|_{\mathcal{T}=\mathcal{T}_c} \sim (\mathfrak{h} - \mathfrak{h}_c)^{1/\delta}$. The exponents, as well as the nonuniversal amplitudes, can be calculated by the standard procedure of expanding f up to third order near the critical point [7], which yields the classical mean field exponents $\beta = 1/2$, $\gamma = 1$, and $\delta = 3$, expectedly, since mean field theory applies to our system.

The fluctuation-dissipation relations naturally hold in the laser mode system; it follows from Eq. (4) that

$$\text{Var}(\psi) = \frac{2}{N}\frac{\partial\langle\psi\rangle}{\partial h} = \frac{2}{N}\chi; \quad (13)$$

that is, the critical exponent γ also describes the divergence of pulse power fluctuations near the critical point. To study the fluctuations *at* the critical point we turn back to Eq. (8), which in the present context is a probability distribution for the single pulse amplitude. Letting $x = P^{-1/2}|\psi|$, the criticality condition implies that for $T = T_c$, $h = h_c$,

$$\mathsf{P}(x) \sim e^{N[a(x-x_c)^4 + O(x-x_c)^5]}, \quad (14)$$

where a is $O(1)$. The fluctuations distribution is non-Gaussian, and the *scale* of the critical fluctuations is $O(N^{-1/4})$, stronger by a factor of $N^{1/4}$ than normal fluctuations. The critical fluctuations are much larger than the typical amplitude of the continuum background, $O(N^{-1/2})$. It follows that the fluctuations of the continuum background are *correlated,* being the SLD analog of the critical *opalescence* phenomenon.

The thermodynamics with $n > 1$ is qualitatively different. The external injection encourages the formation of n equal pulses, clashing with the tendency of the saturable absorber to form a single strong pulse. As a result, the phase diagram consists of an unequal pulse phase for weak noise and weak injection, and an equal pulse phase for strong noise or strong injection. As the two phases are characterized by different symmetries, there can be no smooth transition between them, and they are separated by a phase transition line. However, the phase transition may be continuous or first order, depending on whether the transition order parameter $q = (x - y)/\sqrt{x^2 + y^2}$ is continuous or jumps to a nonzero value at the transition.

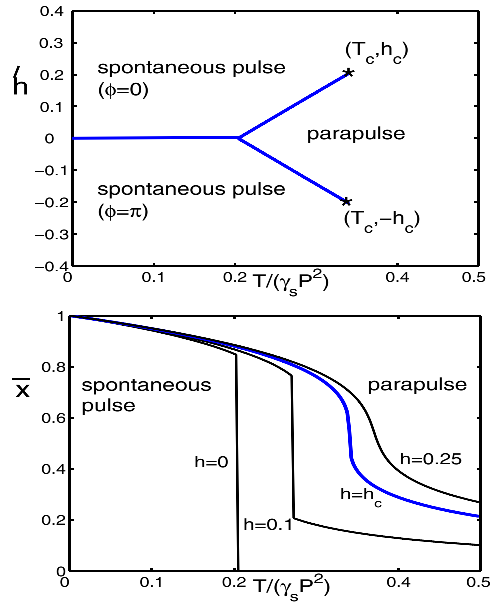

FIG. 1 (color online). The phase diagram for $n = 1$ (homogeneous injection to all modes that matches the cavity repetition rates). Upper figure: The lines give the first order phase transition curve, which terminate at the critical points. The optical phase ϕ follows the phase of $\mathfrak{h}$. Lower figure: Pulse power vs $\mathcal{T}$ for different values of external injection $\mathfrak{h}$.

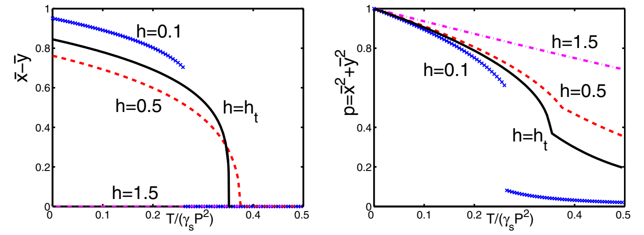

FIG. 2 (color online). $(\bar{x} - \bar{y})$ (left) and $\bar{x}^2 + \bar{y}^2$ (right) vs the normalized temperature for different values of $\mathfrak{h}$ for injection repetition rate twice the laser repetition rate. The nature of the phase transition changes at $\mathfrak{h} = \mathfrak{h}_t$.

We consider in detail the case $n = 2$ where both behaviors occur. It is more convenient to express f of Eq. (10) in terms of q and $p = x^2 + y^2$

$$f(p, q) = -\frac{p^2}{2}(1 + 2q^2 - q^4) - \mathfrak{h}\sqrt{2p\left(1 - \frac{q^2}{2}\right)} - \mathcal{T}\log(1 - p) \quad (15)$$

to be minimized over p and q. The values of the minimizers $\bar{p}$ and $\bar{q}$, giving their "thermal" averages vs $\mathcal{T}$, are given in Fig. 2. f is manifestly symmetric in q, since the pulse amplitudes x and y play symmetric roles, from which it follows that f is always stationary with respect to q when $q = 0$. The condition $\partial_q f = 0$ has another solution $p^3 = \frac{\mathfrak{h}^2}{4}(q^2 - 1)^2(2 - q^2)$, and the global minimum of f is reached in one of these configurations, the first corresponding to equal and the second to unequal pulse mode locking.

Straightforward analysis shows that for large $\mathcal{T}$ the function f has a single minimum at $q = 0$. For large $\mathfrak{h}$ this situation persists as $\mathcal{T}$ is lowered until at $\mathcal{T} = \mathcal{T}_b \equiv \frac{3}{4}\mathfrak{h}^{2/3}(2 - \mathfrak{h}^{2/3})$ the minimum becomes a saddle and two minima with nonzero q form; i.e., q undergoes a continuous phase transition (see Fig. 3). For small $\mathfrak{h}$, however, nonzero q minima appear for $\mathcal{T} > \mathcal{T}_b(\mathfrak{h})$, and at $\mathcal{T}_1(\mathfrak{h})$ exchange stability with the $q = 0$ minimum in a first order phase transition, also shown in Fig. 3.

The intersection of the line of first order phase transition $\mathcal{T}_1(\mathfrak{h})$ and the line of continuous phase transition $\mathcal{T}_b(\mathfrak{h})$ can be shown to occur at $(\mathcal{T}_t, \mathfrak{h}_t) = (\frac{3^{3/2}}{16}, \frac{45}{128})$. $(\mathcal{T}_t, \mathfrak{h}_t)$ is a *tricritical* point [8], with symmetric tricritical phenomena, and the phase diagram Fig. 3 is quite similar to that of metamagnets [12], where tricritical behavior is known to occur. In particular, we may define tricritical exponents such as β_t and β_{2t} associated with nonsymmetric (e.g., q) and symmetric (e.g., p) fields, respectively, by $q \sim (\mathcal{T}_t - \mathcal{T})^{\beta_t}$ and $p - p_t \sim (\mathcal{T}_t - \mathcal{T})^{\beta_{2t}}$ near the tricritical point; see Fig. 2. As before, the exponents take the classical values $\beta_t = 1/4$ and $\beta_{2t} = 1/2$ [8]. Near the continuous phase transition line there are ordinary critical phenomena, for example, $q \sim (\mathcal{T}_b - \mathcal{T})^{\beta}$ and $p - p_b \sim (\mathcal{T}_b - \mathcal{T})^{\beta_2}$, where $\beta = 1/2$ and $\beta_2 = 1$. At the tricritical point fluctuations are enhanced by a factor $N^{1/3}$.

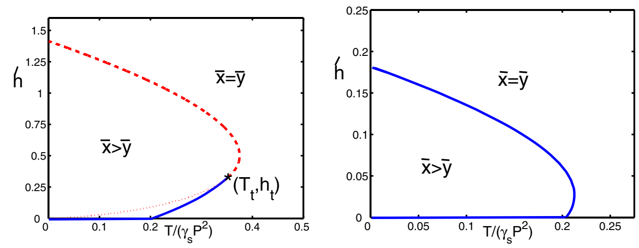

FIG. 3 (color online). The $\mathfrak{h} - \mathcal{T}$ phase diagrams for repetition rate ratios $n = 2$ (left) and $n = 20$ (right). The bold line is a first order phase transition curve. The dotted and dashed lines form together the bifurcation curve $\mathcal{T}_b(\mathfrak{h})$, of which the first is a continuous phase transition line.

When the ratio of the repetition rates n is three or larger, one can show that the transition between the equal and unequal pulse phases is always first order; a typical phase diagram is shown in the right panel of Fig. 3. Critical and multicritical phenomena can be observed in these cases under an external injection of unequal pulses.

The continuation of the phase transition line under external injection can be meaningful for lasers that can remain *metastably* mode locked for an exponentially long lifetime [4].

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